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Mass Customization and the "Parts Capacity-Planning Problem"

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We study a new parts capacity planning problem motivated by a global auto manufacturer that offers 100-500 car options. Technological considerations result in a set of engineering constraints that determine which subsets of options can be combined to produce a car. The set of end-products (producible configurations) is in the order of 10^{25}-10^{40}. Since it is impossible to forecast the demand for producible configurations, firms forecast demand for options. Parts' requirement cannot be directly determined based on options' forecast since a large number of parts' requirements (up to 60%) is based on the combinations of options selected. The options' forecast does not map into a unique configurations-level forecast. Furthermore, given that parts' requirement depends on the configurations sold, the options' forecast implies a range for the parts' requirement. The problem of determining the ranges for parts' requirement is a large-scale NP-Hard problem. We develop an effective approach for solving large industrial instances and compare our approach to that of the current practice.

Our approach provides accurate ranges that are essential for developing procurement contracts. If the firm is able to tighten the ranges then it can lower procurement costs. Therefore, we identify how additional forecasting data can be used to tighten the ranges. We finally present two important extensions. First, we illustrate how our methods can be used when the initial forecast for options' demand are ranges and not point estimates. Second, it is common for several parts to be procured from the same supplier. We show how to estimate joint ranges for multiple parts.

Subject classifications: Mass customization. Parts capacity management.

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1. Introduction

Our problem is motivated by a global auto manufacturer (GAM) that allows customers to configure their cars online. Automobiles consist of a number of options such as engines, transmissions, chassis, electronic systems, interior and exterior designs, and suspensions. Customers configure end-products, also called *producible configurations*, by choosing options that are compatible (see for example Ulrich (1994), Walker and Bright (2013), Feitzinger and Lee (1997), Fohn et al. (1995)). For a general discussion on mass customization, configurable products, and their literature reviews, see Pine (1993), Da Silveira et al. (2001), Heiskala (2007), Sabin and Weigel (1998). The GAM offers 100-500 options for a *car model*, resulting in 10^{25}-10^{40} producible configurations (configurations that are *engineering*-wise producible).

Traditionally, manufacturers have relied on demand forecasts at the level of configurations for parts-capacity planning, production planning, supplier contracts, and pricing decisions (Hax and Candea 1984, Orlicky 1974, Whybark and Williams 1976). In mass customization, it is impossible to forecast configurations' demand because of extremely large number of configurations. Typically, firms first forecast the cumulative demand—i.e., the total number of cars sold over the planning horizon. Second, they forecast options' demand in the form of an options-level penetration statistic (OPS). An OPS consists of a penetration rate for each of the options being offered (the penetration rate of an option represents the fraction of cars that they believe will have that option).

Our analysis on the data received from the GAM indicates that 30-60% of the parts required to produce a car model depend on the combination of options used (we refer to parts, components, and subassemblies as *parts*). For example, a specific combination of an engine and a transmission type generates a number of additional parts. For an industrial instance that has 433 options, Fig. 1 shows how many parts are defined by how many options. In this instance, 63% of parts' requirement depend on a single option and 37% of the parts' requirement depend on multiple options.

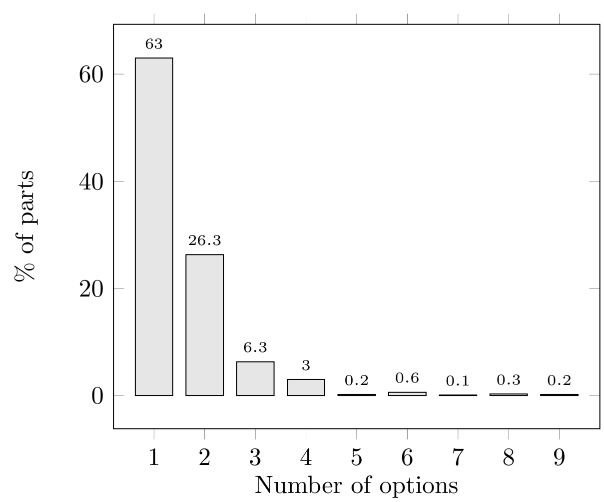

Figure 1 Percentage of parts that are defined based on different number of options.

In short, we need configurations-level forecast to estimate parts requirements. However, the relationship between an OPS and a configurations-level demand is a point-to-set mapping, as shown in the following example.

EXAMPLE 1. **(Point-to-Set Mapping)** Consider two options A' and A'' with penetration rates of 0.5 for both. Assume that a car can have both or one or none of these two options. This results in the following producible configurations: $\{\}$, $\{A'\}$, $\{A''\}$, and $\{A', A''\}$. Observe that the following configurations-level demands are consistent with the given OPS: (i) $0.5\{\} + 0.5\{A', A''\}$, (ii) $0.5\{A'\} + 0.5\{A''\}$, or (iii) $0.25\{\} + 0.25\{A'\} + 0.25\{A''\} + 0.25\{A', A''\}$. □

Similar to OPS, we let a configurations-level penetration statistic (CPS) consist of a penetration rate (fraction of the demand) for each producible configuration. A configuration is producible if it satisfies engineering constraints. Note that a CPS is a convex combination of producible configurations. A CPS is (*marketing*-wise) *consistent* if it satisfies the given OPS. As shown in Example 1, a multitude of CPSs are consistent with the given OPS. Each CPS maps to a specific quantity for a part's requirement. Thus, a given OPS maps to a range for a part's requirement. In this paper, we will also express a part's requirement in terms of a requirement rate. A part's requirement is the ratio of the required quantity of that part to the total demand over the planning horizon.[1]

The GAM procures parts from approximately 20,000 suppliers and it is estimated that parts' cost constitutes approximately two-thirds of the total cost, or about \$90 billion per year. An important element of the procurement contracts is the ranges of parts' requirement. Inaccurate calculation of these ranges has resulted in significant excess inventory and shortages of parts in the GAM. In addition, the wider the ranges, the costlier the parts' capacity acquisition (Bassok and Anupindi 2008). Therefore, it is essential to find these ranges as precisely and as narrowly as possible.

In Example 1, we assumed that any combination of the two options can be selected. However, in general, this is not the case. A set of *engineering rules* exists that determines the compatibility of options. For example, some options are mutually incompatible while, in other instances, selection of an option may require selection of another option. These considerations result in a set of complex engineering constraints, which we refer to as *rules* (Fattahi et al. 2017). The following example illustrates the rules and is referred to throughout the paper.

EXAMPLE 2(a). (Fig. 2(a)) In this example, there are 10 options which we label as $A_1, \ldots, A_{10}$. These options can be engine types, transmissions, body types, sunroofs, audio systems, and so forth. Six of these ten options have assigned penetration rates (denoted by $p(A_3)$, $p(A_4)$, and so forth)—e.g, the penetration rate of A_3 is 0.019 meaning that it is believed that 1.9% of the planning horizon demand will have option A_3. There are 20 rules. Notations "$\neg$", "$\wedge$", "$\vee$", and "$\Rightarrow$" mean "*negation*," "*and*," "*or*," and "*implies*," respectively. For example, the first rule ($A_1 \Rightarrow A_2 \vee A_5 \vee \neg A_{10}$) means that if option A_1 is selected, then either A_2 must be selected or A_5 must be selected, or A_{10} must not be selected. □

Figure 2(a):
Set of options:
$\{A_1, A_2, A_3, A_4, A_5,$
$A_6, A_7, A_8, A_9, A_{10}\}$.
Options with penetration rates:
$p(A_3) = 0.019$
$p(A_4) = 0.861$
$p(A_6) = 0.019$
$p(A_7) = 0.050$
$p(A_8) = 0.046$
$p(A_{10}) = 0.385$.
Rules for selecting options:
$A_1 \Rightarrow A_2 \vee A_5 \vee \neg A_{10}$
$A_1 \Rightarrow A_3 \vee \neg A_7 \vee \neg A_8$
$A_1 \Leftarrow A_7 \wedge A_6 \wedge \neg A_4$
$A_2 \Rightarrow A_3 \vee A_5 \vee A_6 \vee \neg A_7 \vee \neg A_8$
$A_2 \Leftarrow A_4 \wedge A_5 \wedge A_6 \wedge A_7$
$A_2 \Leftarrow \neg A_8 \wedge \neg A_9 \wedge \neg A_{10}$
$A_3 \Rightarrow A_1 \vee A_2 \vee \neg A_4 \vee \neg A_5$
$A_4 \Leftarrow A_1 \wedge A_2 \wedge A_3 \wedge A_5$
$A_5 \Rightarrow A_7 \vee A_8$
$A_5 \Rightarrow A_1 \vee A_2 \vee A_3 \vee A_4 \vee A_6 \vee A_9$
$A_6 \Rightarrow A_5$
$A_7 \Rightarrow A_1 \vee A_5 \vee A_4$
$A_8 \Rightarrow A_9 \vee A_{10}$
$A_8 \Rightarrow A_1$
$A_9 \Rightarrow A_{10}$
$A_9 \Rightarrow A_{10} \vee A_7$
$A_9 \Leftarrow A_1 \wedge A_2$
$A_{10} \Rightarrow A_1$
$A_{10} \Rightarrow A_2 \vee A_3 \vee A_4 \vee A_5$
$A_{10} \Leftarrow A_1 \wedge A_2$.

Figure 2(b):
Set of parts:
$\{R_1, R_2, R_3\}$.
Condition codes:
$A_1 \longrightarrow 1 \times R_1$
$(A_2 \wedge \neg A_3) \vee (A_2 \wedge A_4) \longrightarrow 2 \times R_1$
$(\neg A_5 \wedge A_7) \longrightarrow 4 \times R_1$
$(A_3 \wedge A_4) \vee (A_3 \wedge A_6) \longrightarrow 1 \times R_2$
$(A_1 \wedge \neg A_2 \wedge A_3) \longrightarrow 2 \times R_3$
$A_3 \vee (A_4 \wedge \neg A_5 \wedge \neg A_6) \longrightarrow 1 \times R_3$.

Figure 2 Inputs of our problem for Example 2.

A subset of options that satisfies all rules define a *producible configuration*. In the example above, $\{A_1, A_4, A_5, A_6, A_8, A_{10}\}$ and $\{A_2, A_4\}$ are examples of producible configurations. In general, finding a producible configuration is the well-known NP-complete Satisfiability problem (Garey and Johnson 2002).

The relationship between configurations and parts' requirement is mediated by *condition codes*. Condition codes are stated in terms of associated options. Observe that in Fig. 2(b), there are 6 condition codes that define the requirement for parts R_1, R_2, and R_3. Notation "$\longrightarrow$" means that if the left hand side is true for a given configuration, then the number of parts specified on the right-hand side are required. For example, the third condition code ($(\neg A_5 \wedge A_7) \longrightarrow 4 \times R_1$) means that if option A_5 is not selected and option A_7 is selected, then we require 4 units of part R_1. The total part requirement is based on all condition codes that hold.

For the producible configuration $\{A_1, A_4, A_5, A_6, A_8, A_{10}\}$, the requirement for part R_1 is computed based on the three condition codes (Fig. 2(b)), and since only one holds, then one unit of R_1 is needed. Similarly, for the other producible configuration $\{A_2, A_4\}$, we need two units of R_1.

We define parts capacity-planning problem (PCPP) as the problem of finding the ranges for parts' requirement. This problem consists of the following main inputs: (i) forecast OPS, (ii) *rules* that define producible configurations, and (iii) *condition codes* that relate options to the parts' requirements. To the best of our knowledge, this problem is new and has not been studied in the literature. Our contributions are as follows.

1. Traditionally, parts' requirements have been computed using configurations-level demand forecast and bill of materials. We present a new problem and develop a model that finds ranges for parts' requirements where instead of configurations-level forecast and bill of materials, only *options-level penetration rates*, *rules*, and *condition codes* are available. Our formulation is easy to understand, as it intuitively captures the rules and condition codes.

2. PCPP is a challenging large-scale NP-hard problem. We develop a methodology that involves first reformulating PCPP with the following desirable properties: (i) the objective function is *convex*, and (ii) the feasible set is only the *convex hull of producible configurations*. Second, our methodology sequentially construct the feasible set until an optimal solution is found. We show that our approach effectively solves large industrial instances.

3. We compare our approach to the one used in GAM. The proposed approach provides accurate ranges for parts' requirements while our experiments on an industrial instance indicate that the existing practice has a *mean absolute percentage error* of 35.9%. This can result in substantial *shortage* and/or *excess* inventory of parts.

4. The range for a part's requirement, in principle, can be reduced if we can obtain additional information on demand. We address the following questions. What if the firm can forecast the penetration rates for combination of options, which combination should it forecast? And what will be the impact of additional information on the range?

5. Finally, we extend our approach to consider the following cases: (i) options' penetration rates are given as ranges, and (ii) the firm requires to find a range for the joint requirement of multiple parts.

In short, we develop a new methodology that helps procurement managers in industries with highly configurable products. Our approach provides accurate estimates of the ranges of parts' requirements, which results in substantial reduction in parts' costs and wastage. We finally note that this problem is not limited to the automotive industry. In fact, other industries including, but not limited to, consumer electronics, aircraft, and computer industries offer configurable products (Feitzinger and Lee 1997, Fohn et al. 1995, Kristianto et al. 2015, Rodriguez and Aydin 2011).

The remainder of the paper is organized as follows. In Section 2, we develop a simpler representation of condition codes and formulate the PCPP as a mixed-integer linear programming (MILP) problem. Section 3 presents our methodology for solving industrial instances in a reasonable amount of time and a lower bound on the optimal range. In Section 4, we show how and which additional information regarding penetration rates of single and joint options can assist in narrowing the ranges and then propose a heuristic guideline for selecting which additional information to acquire. In Section 5, we apply our methodology to an industrial instance given to us by the GAM, provide a detailed description of the approach currently used by the GAM, and present a numerical comparative analysis. Section 6 extends and generalizes our methodology to solve two important variations of PCPP. Finally, Section 7 summarises the proposed methodology, discusses some of the limitations, and provides directions for future research.

2. Parts capacity-planning problem (PCPP) formulation

In this section, we formulate the PCPP as a mixed-integer linear programming (MILP) problem that intuitively captures the engineering rules, options' forecast, and condition codes. As stated earlier, rules determine whether a potential configuration (a subset of options) is producible. The problem of determining if there exists a producible configuration is the well-known satisfiability problem. Therefore, it is not practical to explicitly compute all producible configurations. Instead, we implicitly characterize the set of all CPSs that satisfy the given OPS. We refer to CPSs that satisfy the given OPS as *consistent* CPSs. We then map these consistent CPSs to the parts requirements through condition codes. Each CPS maps to a value for each part's requirement. Since there is a set of consistent CPSs (as opposed to a single CPS), we obtain a range for each part's requirement. To find the minimum (respectively, maximum) value of the range, we formulate a problem that, among the set of all consistent CPSs, finds a CPS that minimizes (respectively, maximizes) the part's requirement. In short, this problem has two sets of constraints: (i) configurations have to be producible meaning that they have to satisfy the *rules*, which we refer to as *engineering constraints*, and (ii) CPSs have to satisfy the given OPS, which we refer to as *marketing constraints*.

We use letter A to denote options, letter R to denote parts, and letter F to denote propositional formulas (propositional formulas are used in the representation of rules and condition codes). Let $\mathcal{N}$, $\mathcal{R}$, and $\mathcal{F}$ denote the set of all options, parts, and propositional formulas, respectively. We also use index $i = 1, \dots, n$ for options, where $n := |\mathcal{N}|$.

Condition codes determine part requirements for configurations and are stated in the form of $F \longrightarrow \alpha R$, meaning that if the propositional formula F is "true," we then require α units of part R. We define $\mathscr{C}(R) := \{(F, \alpha) \in \mathcal{F} \times \mathbb{Z}_{++} | F \longrightarrow \alpha R\}$, as the set of all condition codes for part R. For example, for part R_2 in Fig. 2, we have: $\mathscr{C}(R_2) = \{((A_3 \wedge A_4) \vee (A_3 \wedge A_6), 1)\}$.

Let $\mathbf{y}$ denote a generic producible configuration and $\mathbb{Y}$ denote the set of all producible configurations. We show how to determine a part's requirement for configuration $\mathbf{y}$ using the condition codes. Recall that associated with each configuration is a set of options. To build configuration $\mathbf{y}$, we require $\sum_{(F,\alpha) \in \mathscr{C}(R)} \alpha v_{\mathbf{y}}(F)$ units of part R, where $v_{\mathbf{y}}(F)$ is the value of formula F in configuration $\mathbf{y}$, i.e., if $\mathbf{y}$ satisfies F, then $v_{\mathbf{y}}(F) = 1$, and otherwise $v_{\mathbf{y}}(F) = 0$. For example in Fig. 2, the number of units of part R_1 that is required to produce configuration $\mathbf{y}$ is equal to: $v_{\mathbf{y}}(A_1) + 2v_{\mathbf{y}}((A_2 \wedge \neg A_3) \vee (A_2 \wedge A_4)) + 4v_{\mathbf{y}}(\neg A_5 \wedge A_7)$. Observe that the subset $\{A_2, A_4\}$ is a producible configuration. Part R_1's requirement for this configuration is equal to $0 + 2 \times 1 + 4 \times 0$; hence, producing this configuration requires 2 units of part R_1.

Let $p(A)$ denote the penetration rate of option A and $p(F)$ denote the penetration rate of formula F (note that penetration rates of formulas will be used in characterizing parts' requirement). For example, if $F = A_1 \wedge A_2$, then $p(A_1 \wedge A_2)$ is the joint penetration rate of $A_1 \wedge A_2$ and it shows the fraction of cars sold over the planning horizon that include both options A_1 and A_2. We sometimes use p_A instead of $p(A)$ for ease of notation.

Recall that a convex combination of producible configurations represents a CPS. Thus, a CPS consists of a coefficient $a_{\mathbf{y}}$ for each producible configuration $\mathbf{y} \in \mathbb{Y}$ such that these coefficients satisfy: $a_{\mathbf{y}} \geq 0$, for all $\mathbf{y} \in \mathbb{Y}$, and $\sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} = 1$. A CPS maps to a specific value for a part's requirement. Let Q_R denote the requirement of part R for a given CPS. We have:

$$Q_R = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha v_{\mathbf{y}}(F) = \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} v_{\mathbf{y}}(F) = \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha p(F).$$

The first equality follows from the definition of Q_R. Note that $\sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} v_{\mathbf{y}}(F)$ is the fraction of cars that satisfy formula F; hence, the last equality holds because $p(F) = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} v_{\mathbf{y}}(F)$.

Since there are many consistent CPSs and each map into a specific value for part R's requirement, a range is obtained for Q_R. This range is determined by minimizing and maximizing Q_R over all consistent CPSs. Hence, we use $Q_R = \sum_{(F,\alpha)\in\mathscr{C}(R)} \alpha p(F)$ as the objective function of our MILP model. The difficulty, however, is that some of the F's in this equation may be complex. Thus, we perform the following simplification. If $(F,\alpha) \in \mathscr{C}(R)$ and F is complex, then we define a new *artificial* option A' and replace (F,α) with (A',α) in the set $\mathscr{C}(R)$, while adding $F \Leftrightarrow A'$ to the set of rules. For example, if $(A_3 \wedge A_4, 2) \in \mathscr{C}(R)$, then we introduce a new option A', replace $(A_3 \wedge A_4, 2)$ with $(A', 2)$, and add $A_3 \wedge A_4 \Leftrightarrow A'$ to the set of rules. Let $\tilde{\mathscr{C}}(R)$ denote the simplified version of the set $\mathscr{C}(R)$. Hence, our objective function becomes $Q_R = \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i$.

We next formulate the feasible region of our problem. We start by formulating the rules. The GAM's *engineering* rules are a set of well-defined propositional formulas and can be written in *conjunctive normal form* (CNF) (see, for example, Tseitin (1968) and Wilson (1990)). A CNF formula is in the form of $C_1 \wedge \cdots \wedge C_\ell$ where $C_1, \ldots, C_\ell$ are disjunctive clauses. A clause is called "disjunctive" if it is the disjunction of some literals (a literal is either an option or its negation). Although our methodology can be easily extended to rules in any format, we assume rules are in CNF.

We write one constraint for each clause.[2] As an example, consider clause $A_1 \vee \neg A_2$. This clause is formulated as $y_{A_1} + (1 - y_{A_2}) \geq 1$, where binary variables y are defined as follows: $y_A = 1$ if option A is selected, and $y_A = 0$ otherwise. This constraint means that either $y_{A_1} = 1$ or $y_{A_2} = 0$ or both. Hence, we can formulate the set of all producible configurations $\mathbb{Y}$ as follows:

$$\mathbb{Y} := \{\mathbf{y} \in \{0,1\}^n | \sum_{i\in PO_C} y_i + \sum_{i\in NO_C} (1 - y_i) \geq 1, \ \forall C\},$$

where, $\mathbf{y} := (y_1, \ldots, y_n)$, and PO_C (respectively, NO_C) is the set of positive (respectively, negative) options in clause C. For example, if $C = \neg A_3 \vee A_4$, then $PO_C = \{A_4\}$ and $NO_C = \{A_3\}$. In this paper, we assume $\mathbf{0} \in \mathbb{Y}$.

An OPS consists of a penetration rate $p_i \in [0,1]$ for each option $i = 1, \ldots, n$. Let $\mathbf{p} = (p_1, \ldots, p_n)$ denote an OPS. This OPS satisfies the engineering rules if there exists a CPS, $a_{\mathbf{y}}$'s, such that $\sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} y_i = p_i$, for all i. In other words, an OPS $\mathbf{p}$ satisfies the engineering constraints if $\mathbf{p} \in conv(\mathbb{Y})$, where $conv(\mathbb{Y})$ denotes the convex hull of $\mathbb{Y}$. We refer to the condition $\mathbf{p} \in conv(\mathbb{Y})$ as the *engineering constraints*. Finally, we note that the *marketing* constraints (forecast OPS) are formulated as: $p_i = \hat{p}_i$, for all $i \in \mathcal{N}_1$, where $\hat{p}_i$ denotes the forecast penetration rate of option i, and $\mathcal{N}_1$ denotes the set of options for which forecast penetration rates exist. For example in Fig. 2, $\mathcal{N}_1 = \{A_3, A_4, A_6, A_7, A_8, A_{10}\}$. Thus, we formulate our problem, which we refer to as (P1), as follows:

$$\text{(P1):}\quad \min/\max\ \ Q_R = \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \tag{1}$$

$$\text{s.t.}\quad p_i = \hat{p}_i,\ \ \forall i \in \mathcal{N}_1, \tag{2}$$

$$\mathbf{p} \in conv(\mathbb{Y}). \tag{3}$$

Let $Q^*_{R,L}$ and $Q^*_{R,U}$ respectively denote the optimal values of the minimization (P1min) and maximization (P1max) problems. The detailed MILP formulation of (P1) along with an example is given in Appendix A.1. Last, we establish the NP-hardness of (P1).

Proposition 1 **(NP-Hardness)**. *Problem (P1) is NP-hard.*

The proof simply follows from noting that satisfiability problem (known to be NP-complete; see, e.g., Garey and Johnson (2002)) is imbedded in (P1).

3. Solution approach

PCPP is a large-scale optimization problem and existing approaches for solving the MILP representation of this problem are not effective. Therefore, we need to develop a specialized solution approach. In this section, we propose a methodology that involves "dualizing" the *marketing* constraints so that the new formulation is equivalent to the original one and has the following features: (i) the feasible region is only $conv(\mathbb{Y})$, which is a polytope and its extreme points are producible configurations, and (ii) the objective function is piecewise linear and convex. We sequentially construct the feasible set until an optimal solution is found. This feature of our solution approach is similar to a variant of the Frank-Wolfe method, which has been shown to be effective for solving a class of large convex optimization problems (Bach 2013, Clarkson 2010, Jaggi 2013). Finally, we present a procedure to find a bound on the optimal range which can assist in obtaining an optimality gap for the early termination of the algorithm.

3.1. An equivalent formulation of PCPP

Recall that (P1) minimizes (respectively, maximizes) $Q_R = \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i$ subject to *engineering* and *marketing* constraints. To work with a more manageable feasible region, we relax the *marketing* constraints and add penalty terms for violation of these constraints to the objective function. This changes the objective function to $\sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \pm \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i|$, where M is a sufficiently large number. In this subsection, we show that this objective function for the minimization and maximization cases can be transformed to $\|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p}-\hat{\mathbf{p}}_{R,L})\|_1$ and $\|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p}-\hat{\mathbf{p}}_{R,U})\|_1$, respectively, for carefully chosen vectors $\mathbf{w}_R$, $\hat{\mathbf{p}}_{R,L}$, and $\hat{\mathbf{p}}_{R,U}$, where $\|.\|_1$ indicates the first norm, $\mathbf{Diag}(\mathbf{w}_R)$ is a diagonal matrix with $\mathbf{w}_R$ as the diagonal elements, and $\mathbf{p}$ is the vector of decision variables (all vectors are in $\mathbb{R}^n$). One could view $\hat{\mathbf{p}}_{R,L}$ and $\hat{\mathbf{p}}_{R,U}$ as the target points for $\mathbf{p}$ where the objective is to push $\mathbf{p}$ as close as possible to these target points. Moreover, $\mathbf{w}_R$ could be viewed as a weight vector that determines proportional importance for different options—i.e., the larger the weight of an option, the more effort should be made for matching the corresponding decision variable p with its target value.

Define vector $\mathbf{w}_R = (w_{R,1},\ldots,w_{R,n}) \in \mathbb{R}^n$ as follows: for all $i\in\mathcal{N}$, if $i\in\mathcal{N}_1$, then let $w_{R,i} := M$; otherwise, let $w_{R,i} := \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha$. Define vectors $\hat{\mathbf{p}}_{R,L}$, $\hat{\mathbf{p}}_{R,U} \in \mathbb{R}^n$ as follows: for all $i\in\mathcal{N}$, if $i\in\mathcal{N}_1$, then let $\hat{p}_{R,L,i} := \hat{p}_i$ and $\hat{p}_{R,U,i} := \hat{p}_i$; otherwise, if $i\in\mathcal{N}\backslash\mathcal{N}_1$, then let $\hat{p}_{R,L,i} := 0$ and $\hat{p}_{R,U,i} := 1$. For example, for part R_1 in Example 2, we have: $\mathbf{w}_{R_1} = (1,0,M,M,0,M,M,M,0,M,2,4)$, $\hat{\mathbf{p}}_{R_1,L} = (0,0,0.019,0.861,0,0.019,0.050,0.046,0,0.385,0,0)$, and $\hat{\mathbf{p}}_{R_1,U} = (1,1,0.019,0.861,1,0.019,0.050, 0.046,1,0.385,1,1)$.

Our equivalent formulation can be expressed as the following problems, referred to as (P2L) and (P2U) (note that (P2) refers to both problems (P2L) and (P2U)):

$$\text{(P2L)} \ : \ \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p}-\hat{\mathbf{p}}_{R,L})\|_1,$$
$$\text{(P2U)} \ : \ \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w}_R)(\mathbf{p}-\hat{\mathbf{p}}_{R,U})\|_1.$$

Theorem 1 shows that solving (P2) provides an optimal solution to (P1).

THEOREM 1 **(Nonlinear Programming Equivalence)**. *If M is sufficiently big, then an optimal solution of problem (P2) is also an optimal solution for problem (P1).*

Proof is given in Appendix B.1. Problem (P2) has the following features: (i) the feasible region is the convex hull of $\mathbb{Y}$, (ii) the objective is to minimize the sum of weighted absolute differences between p_i's and $\hat{p}_{R,L,i}$'s (or $\hat{p}_{R,U,i}$'s), and (iii) the objective functions are piecewise linear and convex. Finally, we remark that our approach is different from the classical *Lagrangian Relaxation* method in two aspects: (a) M is a scalar and not a vector of Lagrange multipliers, and (b) this approach does not result in any optimality gap as shown in Theorem 1. In the remainder, we propose an approach for solving problem (P2).

3.2. Applying Frank-Wolfe for solving problem (P2)

We now discuss how to solve problem (P2L), and the same approach can be used for solving problem (P2U). Our approach is given in Algorithm 1, which is known as the "fully corrective" variant of the Frank-Wolfe method (Bach 2013, Clarkson 2010, Jaggi 2013). Note that the index for the parts (R) is dropped in the remainder of this subsection for ease of notation.

At each iteration $k \geq 1$, we find the best known solution $\mathbf{p}^{(k)}$ by minimizing the objective function over the convex hull of the set $\{\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k-1)}\}$, as shown in Line 3 of Algorithm 1. This is an easy optimization problem with linear constraints and a piecewise linear and convex objective function.

Since the objective function of (P2L) is piecewise linear and convex, we use the subgradient vector $\mathbf{g}^{(k)} \in \mathbb{R}^n$ at iteration $k \geq 1$, which is defined as follows:

$$g_i^{(k)} := \begin{cases} w_i & , \text{ if } p_i^{(k)} > \hat{p}_{L,i} \\ -w_i & , \text{ if } p_i^{(k)} < \hat{p}_{L,i} \\ \sim \text{Uniform}[-w_i, w_i] & , \text{ if } p_i^{(k)} = \hat{p}_{L,i}, \end{cases} \tag{4}$$

for all $i \in \mathcal{N}$. Note that, if $p_i^{(k)} = \hat{p}_{L,i}$, we can set $g_i^{(k)}$ to any value in $[-w_i, w_i]$; hence, we generate a number using a uniform distribution to allow for finding new points in different directions as the algorithm iterates.

In Line 5, we find a new extreme point of the feasible region *conv*$(\mathbb{Y})$ by minimizing a linear function $\mathbf{g}^{(k)T}\mathbf{y}$. We exclude the previously found solutions using *no-good* constraints (Hooker 2000). For example, if $\mathbf{y}^{(0)} = (y_1^{(0)}, \ldots, y_n^{(0)})$, a no-good constraint to exclude $\mathbf{y}^{(0)}$ is:

$$\sum_{i: y_i^{(0)}=1} y_i - \sum_{i: y_i^{(0)}=0} y_i \leq \left(\sum_{i: y_i^{(0)}=1} 1 \right) - 1.$$

Note that although $\mathbf{y}^{(0)}$ is not feasible for this constraint, any $\mathbf{y} \neq \mathbf{y}^{(0)}$ is feasible. If we generate exactly one extreme point of *conv*$(\mathbb{Y})$ at each iteration, then the optimization problem in Line 5 has exactly k no-good constraints at iteration $k \geq 1$. Using no-good constraints has the following advantages: (i) Algorithm 1 is guaranteed to generate a new extreme point at each iteration, and (ii) if $|\mathbb{Y}|$ is small, Algorithm 1 can explore all points in $\mathbb{Y}$, in which case the problem in Line 5 becomes infeasible and $\mathbf{p}^{(k)}$ is a guaranteed optimal solution of (P2L).

The Frank-Wolfe algorithm converges with $\mathcal{O}(1/k)$ if the feasible region is a compact convex subset of any vector space and the objective function is convex and continuously differentiable (Dunn and Harshbarger 1978, Jones 1992, Patriksson 1993, Clarkson 2010, Jaggi 2013). This convergence rate exists because of the existence of a finite *curvature constant* for the objective function (see for example Jaggi (2013)). However, the objective function of problem (P2) does not have a finite

Algorithm 1 Solving problem (P2L)

Input: Rules, $\mathbf{w}$, $\hat{\mathbf{p}}_L$. ▷ "Rules" characterize $\mathbb{Y}$.

Output: $\mathbf{p}_L^*$. ▷ $\mathbf{p}_L^*$ is the optimal solution of (P2L).

1: Find $\mathbf{y}^{(0)} \in \mathbb{Y}$; ▷ A producible configuration for initialization.

2: **for** $k = 1, 2, 3, \cdots$ **do**

3: Solve $\mathbf{p}^{(k)} := \arg\min_{\mathbf{p} \in conv(\{\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k-1)}\})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_L)\|_1$;

4: Define $\mathbf{g}^{(k)}$ as in Eq. (4);

5: Solve $\mathbf{y}^{(k)} := \arg\min_{\mathbf{y} \in \mathbb{Y} \setminus \{\mathbf{y}^{(0)}, \mathbf{y}^{(1)}, \ldots, \mathbf{y}^{(k-1)}\}} \mathbf{g}^{(k)T}\mathbf{y}$; **if** infeasible **then** let $\mathbf{p}_L^* := \mathbf{p}^{(k)}$; Stop!

6: **end for**.

curvature constant; hence, the convergence rate of Algorithm 1 cannot be guaranteed. Nonetheless if problem (P2) is approximated by using $\|.\|_2^2$ instead of $\|.\|_1$, the convergence rate $\mathcal{O}(1/k)$ is achieved. Note that using $\|.\|_2^2$ results in finding the optimal solution if $\hat{\mathbf{p}}_L$ and $\hat{\mathbf{p}}_U$ are the optimal solutions of (P2L) and (P2U), respectively. However, this is not true in general.

Our algorithm starts with a producible configuration $\mathbf{y}^{(0)}$ (see Line 1). Moreover, note that any set of producible configurations can be used (instead of $\mathbf{y}^{(0)}$) for initialization. Hence, as a warm start, we use the set of producible configurations that is found by applying the algorithm of Fattahi et al. (2017). Their algorithm finds a set of producible configurations such that the forecast OPS belongs to their convex cone. We will show the effectiveness of our approach for solving industrial instances in section 5.

3.3. Bound on the optimal parts' requirement

Our problem, if solved to optimality, provides the tightest range for a part's requirement. A bound on the range may be needed when the algorithm is terminated prior to completion. Recall that our problem has two sets of constraints: *Engineering* and *Marketing*. An intuitive approach for obtaining a bound is to focus on the options that are present in the condition codes of the part under consideration and drop the *marketing* constraints for the other options. Since the number of options that are present in the condition codes of a part is less than 10, we enumerate all producible *scenarios* for those options and develop a *scenario* based formulation that can be solved in a negligible amount of time.

Let OP_R denote the set of options that are present in the condition codes of part R, e.g. in Fig. 2, $\text{OP}_{R_2} = \{A_3,\ A_4,\ A_6\}$. We enumerate all scenarios for the options in OP_R (this is possible since the size of OP_R is usually less than 10, e.g. see Fig. 1). These scenarios are denoted by 0-1 vectors, e.g., $(1, 0, 1)$ means only A_3 and A_6 are selected. For each scenario, we determine whether the rules are satisfiable (all rules and not only the rules that are related to OP_R). Let SC_R denote the set of satisfiable scenarios, also referred to as *sub-configurations*. With some abuse of notation, we denote sub-configurations by $\mathbf{y} \in \mathrm{SC}_R$.

Let $\tilde{\tilde{\mathbf{p}}}$ denote the vector of penetration rates for the options in OP_R. For each sub-configuration $\mathbf{y} \in \mathrm{SC}_R$, let $\varsigma_{\mathbf{y}}$ denote the number of units of part R required to produce one unit of that sub-configuration. We aim to find a convex combination of the sub-configurations that satisfy the penetration rates for the options in OP_R and minimize (respectively, maximize) the part R's requirement. This is achieved by solving the following linear programming problem, referred to as (PB).

$$\begin{aligned}
\text{(PB):}\quad \min/\max\ \ & Q^{\circ}_R = \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}} \varsigma_{\mathbf{y}} \\
\text{s.t.}\ \ & \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}} \mathbf{y} = \tilde{\tilde{\mathbf{p}}}, \\
& \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}} = 1, \\
& a_{\mathbf{y}} \geq 0, \quad \forall \mathbf{y} \in \mathrm{SC}_R.
\end{aligned}$$

Let $Q^{\circ}_{R,L}$ and $Q^{\circ}_{R,U}$ denote the optimal values of the minimization and maximization cases of problem (PB), respectively. In the following, we show $Q^{\circ}_{R,L} \leq Q^{*}_{R,L}$ and $Q^{*}_{R,U} \leq Q^{\circ}_{R,U}$ (recall that $Q^{*}_{R,L}$ and $Q^{*}_{R,U}$ respectively denote the optimal values of (P1min) and (P1max)).

Proposition 2 **(Bound on the Optimal Range)**. *The optimal values of problems (P1) and (PB) satisfy:* $Q^{\circ}_{R,L} \leq Q^{*}_{R,L}$ *and* $Q^{*}_{R,U} \leq Q^{\circ}_{R,U}$.

The proof is given in Appendix B.2. We note that the quality of this bound can be improved by incorporating additional options in OP_R. The more options included in OP_R the closer the sub-configurations to full configurations.[3] To determine which additional option from $\mathcal{N}_1 \backslash \mathrm{OP}_R$ to incorporate, a measure that captures the relative association (e.g. number of times that they appear in the same rules) with the options that are already in OP_R can be used. In section 5, we show the performance of this bound and how it can be improved by considering more options.

4. Value of additional marketing information

In mass customization, one of the major challenges in parts-capacity planning is the need for narrow ranges for parts' requirement. The GAM's contracts with their suppliers are based on these ranges, and the wider the ranges the costlier the parts-capacity acquisition.

The ranges that we obtain for parts' requirements in Section 3 are provably the tightest ranges that one can find using the available information (forecast OPS, rules, and condition codes). These ranges are sometimes wide mainly because the set of configurations-level demand that satisfies the given options-level forecast is very large. Thus, we have to collect additional information to narrow the obtained ranges. The additional information acts as a new constraint that shrinks the feasible set and improves the ranges. We also note that acquiring additional information is costly, and different information sets are obtained at different costs. In this section, we propose an interactive approach that adds one (or more) information at a time until the desired width of a range is obtained. Specifically, we aim to answer the following questions. What is the impact of additional information (e.g., on the penetration rates of single and joint options) on the ranges of parts' requirement? And, which information set is more likely to have the highest impact? For example, what is the impact of the joint penetration rate of engine `XYZ` and transmission `UVW` on reducing the range of a part's requirement?

4.1. Incorporating new information in the PCPP

We present a method for incorporating new information and ensuring the consistency of the new information with the rules and forecast OPS. We call the constraint $p(F)=\gamma$ (penetration rate of formula F is equal to γ) an *information set*, where F is a propositional formula and $0 \leq \gamma \leq 1$. Note that $p(F)=\gamma$ is equivalent to $p_{A'}=\gamma$ and $A' \Leftrightarrow F$, where A' is an *artificial* option. In other words, to add the information set $p(F)=\gamma$ to PCPP, we define a new *artificial* option A', set the penetration rate of option A' to γ, and add $A' \Leftrightarrow F$ to the set of rules.

Note that, it is necessary for $p(F)=\gamma$ to be *consistent* with the engineering and marketing constraints, meaning that adding this information to the PCPP does not make the problem infeasible. In the following lemma, we determine a range for γ such that the new information $p(F)=\gamma$ is consistent.

Lemma 1 **(Consistency of a New Information Set)**. *Let $a_{L,\mathbf{y}}$ and $a_{U,\mathbf{y}}$ denote the optimal coefficients of configuration $\mathbf{y} \in \mathbb{Y}$ in (P1min) and (P1max), respectively. Define $\gamma_{L,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$ and $\gamma_{U,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{U,\mathbf{y}} v_{\mathbf{y}}(F)$. The information set $p(F)=\gamma$ is consistent if $\min\{\gamma_{L,F},\gamma_{U,F}\} \leq \gamma \leq \max\{\gamma_{L,F},\gamma_{U,F}\}$.*

The proof is in Appendix C.1. The condition in Lemma 1 is sufficient but not necessary. We next generalize this notion and present a method for ensuring the consistency of adding a collection of new information sets.

Lemma 2 **(Consistency of a Collection of New Information Sets)**. *Let $a_{L,\mathbf{y}}$ and $a_{U,\mathbf{y}}$ denote the optimal coefficients of configuration $\mathbf{y} \in \mathbb{Y}$ in (P1min) and (P1max), respectively. Let $\tilde{\mathcal{F}} \subseteq \mathcal{F}$. Define $\gamma_{L,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$ and $\gamma_{U,F} := \sum_{\mathbf{y}\in\mathbb{Y}} a_{U,\mathbf{y}} v_{\mathbf{y}}(F)$, for all $F \in \tilde{\mathcal{F}}$. Let $0 \leq \bar{\gamma} \leq 1$. The following set of constraints is consistent:*

$$p(F) = \gamma_{L,F} + \bar{\gamma}(\gamma_{U,F} - \gamma_{L,F}), \quad \forall F \in \tilde{\mathcal{F}}.$$

We skip the proof as it is similar to the proof of Lemma 1. The condition of Lemma 2 is sufficient but not necessary. To illustrate the application of Lemma 2, assume that the allowable ranges for the information on F' and F'', obtained using Lemma 1, are $[0.2, 0.4]$ and $[0.5, 0.9]$, respectively. We can simultaneously add the following two information sets to PCPP: $p(F') = 0.3$ and $p(F'') = 0.7$ (because these values are the midpoints of the allowable ranges, these constraints are consistent).

4.2. Values of different information sets

Having new information sets on different formulas can result in different reductions in the range of a part's requirement. For example, is information set $p(A_1 \wedge A_2)$ better/worse than $p(A_1 \wedge A_2 \wedge A_3)$? What about $p(A_1 \vee A_2)$ versus $p(A_1 \vee A_2 \vee A_3)$? In this subsection, we show that, in general, it is not theoretically possible to establish ordering. Consequently, we present a heuristic guideline for comparing the values of different information sets.

Proposition 3 **(Ordering Different Information Sets)**. *Let $F', F'' \in \mathcal{F}$ and $0 \leq \gamma \leq 1$ such that each of the following information sets is consistent: $p(F') = \gamma$, $p(F' \wedge F'') = \gamma$, and $p(F' \vee F'') = \gamma$. Let $Q^{*'}_{R,L}$ and $Q^{*'}_{R,U}$ denote the optimal values of PCPP if $p(F') = \gamma$ is added, $Q^{*''}_{R,L}$ and $Q^{*''}_{R,U}$ denote the optimal values of PCPP if $p(F' \wedge F'') = \gamma$ is added, and $Q^{*'''}_{R,L}$ and $Q^{*'''}_{R,U}$ denote the optimal values of PCPP if $p(F' \vee F'') = \gamma$ is added. Then,*

(a) if $\gamma = 0$, then $Q^{*''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*'''}_{R,L} \leq Q_R \leq Q^{*'''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*''}_{R,L}$,

(b) if $\gamma = 1$, then $Q^{*'''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*''}_{R,L} \leq Q_R \leq Q^{*''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*'''}_{R,L}$.

Proof is in Appendix C.2. To illustrate Proposition 3, let $F' = A_1 \wedge A_2$ and $F'' = A_3$. Hence, we are interested in comparing the values of information sets $p(A_1 \wedge A_2) = \gamma$ and $p(A_1 \wedge A_2 \wedge A_3) = \gamma$. Note that the former means $100\gamma\%$ of the cars sold over the planning horizon are forecast to have options A_1 and A_2, while the latter means $100\gamma\%$ of the cars sold are forecast to have options A_1, A_2, and A_3. Dependent on the value of γ, one information set can weakly dominate the other.

Proposition 3 implies that if $\gamma = 0$ (or small), then *pooling* of information ($\vee$) is preferred, while if $\gamma = 1$ (or close to 1), then *refinement* of information ($\wedge$) is preferred. However, the value of γ is not known a priori, and hence, the ordering cannot be established in advance.

We next propose a heuristic guideline for comparing two information sets. We prefer F' over F'' if $|\gamma_{U,F'} - \gamma_{L,F'}| > |\gamma_{U,F''} - \gamma_{L,F''}|$, where $\gamma_{U,F'}$, $\gamma_{L,F'}$, $\gamma_{U,F''}$, and $\gamma_{L,F''}$ are calculated as described in Lemma 1.

Information Ordering Criteria. *Information on F' is preferred to information on F'' if $|\gamma_{U,F'} - \gamma_{L,F'}| > |\gamma_{U,F''} - \gamma_{L,F''}|$.*

The intuition for this criteria is as follows: if $p(F')$ changes more than $p(F'')$, it can imply a higher influence on the range, and hence, we expect to see a higher reduction in the range by fixing the value of $p(F')$ compared to $p(F'')$. Our experimental analysis verifies the effectiveness of this criteria. We use this criteria in the design of our interactive approach.

4.3. Interactive approach for reducing the parts' requirement ranges

We develop an interactive approach that consists of an "expert" and a "system." See Appendix C.3 for details and the flowchart. The system solves the PCPP and offers a list of candidate propositional formulas to the expert who then selects one or more of the candidates and determines their penetration rates. The system incorporates this information set and solves the PCPP again. This cycle is repeated until a pre-defined criteria is met. For example, a common criteria used in the GAM is that the width of the range should not be bigger than 20% of the minimum value of the range. The justification is that the supplier can plan for the minimum value of the range and increase the production by working overtime at most 20% of the regular time. An illustrative example is provided in Appendix C.3. We will show how additional information helps narrowing the ranges for parts' requirement on an industrial instance in section 5.

5. Industrial Applications and Computational Experiment

In this section, we first show the computational effectiveness of our approach and the proposed bound on an industrial instance provided by the GAM. Next, we describe the method that is currently used in practice, compare our approach to that of the current practice, and discuss the advantages of our proposed methodology. We finally show the value of additional information on narrowing the parts' requirement ranges.

5.1. Performance of our solution approach on an industrial instance

We show the performance of our algorithm on an industrial instance that we have received from the GAM. This instance has 433 options and 171 rules.[4] In our analysis, we focus on three parts, which we denote by $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$.

We apply Algorithm 1 for solving problem (P2) to obtain ranges on the requirement of the three parts. We implement our algorithm in IBM ILOG CPLEX Optimization Studio 12.6.1 and use a PC with Processor Intel(R) Core(TM) i5-2520M CPU 2.50GHz, 4.00 GB of RAM, and 64-bit Operating System. We run the algorithm for 10,000 seconds and report the results in Fig. 3 for different parts.

We note that major changes occur initially and then the minimum and maximum values of the ranges become approximately flat (except for the minimum value of the range for part $\tilde{R}_3$ that demonstrates large changes around 7,000 seconds). The optimal ranges for parts $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$ are respectively 12.47%, 9.67%, and 9.05% of the minimum values of the optimal ranges (based on the results after 10,000 seconds).

We also apply our approach presented in subsection 3.3 to obtain bounds on the ranges. This results in bounds (2.399,3.100), (0.650,0.800), and (0.800,1.000) for parts $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$, respectively. Fig. 3 shows these bounds using thick dashed lines. The error associated with the minimum

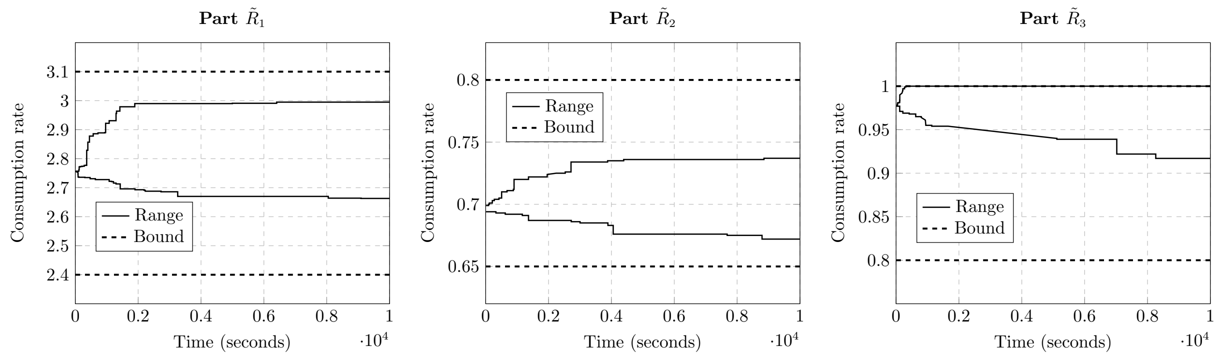

Figure 3 Computing the range using Algorithm 1 and performance of the proposed bound.

values of the bounds are 9.91%, 3.27%, and 12.76%, and with the maximum values of the bounds are 3.5%, 8.55%, and 0%, respectively.

5.1.1. Improving bound on the range As discussed in subsection 3.3, the bound on the ranges can be improved by incorporating additional options in the set OP_R. We next show how such options can be selected and how they impact the bound. We focus on a part $\tilde{R}$ that has 6 options in its condition codes, i.e., $\text{OP}_{\tilde{R}}$ contains 6 options. We construct a graph of *options-relations* where the nodes are the options and there exist an arc between two options if they appear in a same rule. Moreover, the weight of an arc shows the number of times that two options appear in the same rules. Fig. 4 shows a portion of this graph. The nodes with a star (in the left side of the graph: options 25, 49, and 20) are the options that belong to $\text{OP}_{\tilde{R}}$. Among the set of options that are not included in $\text{OP}_{\tilde{R}}$, we choose the one with highest incident weight. In this example, options 26, 44, and 72 are sequentially added to $\text{OP}_{\tilde{R}}$ (note that option 72 is not shown in the graph). Fig. 4 shows improvements on the bound by these additions. The optimal range for this part is also shown by the shaded area. The error of the minimum value of the bound improves from 66.2% to 9.9% and the error of the maximum value improves from 63.6% to 3.5% by adding the three options.

5.2. Description and comparison to current practice

In this subsection, we describe, evaluate, and compare our results to that of the current practice.

5.2.1. Description of the current practice The GAM's approach for determining the requirement of a part R involves generating all producible sub-configurations that include the options that are present in the condition codes of that part. Then, a convex combination of these sub-configurations is found that is as close as possible to the penetration rates of the associated options. This is used to find a point and range estimates for the requirement of part R. Their approach can be formally presented as follows.

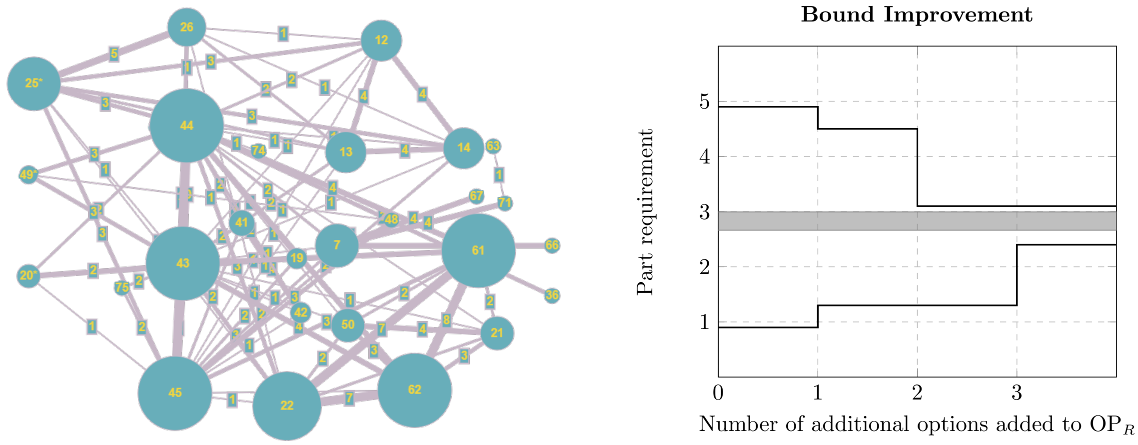

Figure 4 Improving bound by adding new options to OP$_{\tilde{R}}$.

Step 1: Generating sub-configurations. Let OP$_R$ and SC$_R$ be constructed as described in subsection 3.3.

Step 2: Least-squares fit. Recall that $\tilde{\tilde{\mathbf{p}}}$ denotes the vector of penetration rates for the options in OP$_R$. Determine the sub-configurations' coefficients by solving the following quadratic problem:

$$\begin{aligned}
&\min \left\| \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}}\mathbf{y} - \tilde{\tilde{\mathbf{p}}} \right\|_2^2 \\
&\text{s.t.} \sum_{\mathbf{y}\in \mathrm{SC}_R} a_{\mathbf{y}} = 1, \\
&a_{\mathbf{y}} \geq 0, \quad \forall \mathbf{y} \in \mathrm{SC}_R.
\end{aligned}$$

Let $a^*_{\mathbf{y}}$'s denote the optimal solution. In fact, this problem finds a point in the convex hull of SC$_R$ that has the minimum Euclidean distance to $\tilde{\tilde{\mathbf{p}}}$. The optimal point is $\sum_{\mathbf{y}\in \mathrm{SC}_R} a^*_{\mathbf{y}}\mathbf{y}$ that is represented as a convex combination of the sub-configurations.

Step 3: Estimating part R's requirement. Recall that $\varsigma_{\mathbf{y}}$ denotes the number of units of part R required to produce one unit of sub-configuration $\mathbf{y}$. A point estimate of part R's requirement is calculated as $Q^{\diamond}_R := \sum_{\mathbf{y}\in \mathrm{SC}_R} a^*_{\mathbf{y}}\varsigma_{\mathbf{y}}$. A range is then considered for the requirement of part R with a center at $Q^{\diamond}_R$ and a pre-specified radius, e.g., if $Q^{\diamond}_R > 0$, the radius may be specified as 10% of $Q^{\diamond}_R$.

In fact, the optimal point $\sum_{\mathbf{y}\in \mathrm{SC}_R} a^*_{\mathbf{y}}\mathbf{y}$ in step 2 represents a demand realization on sub-configurations, i.e., $a^*_{\mathbf{y}}$ is the fraction of the total demand that includes sub-configuration $\mathbf{y}$. This demand realization maps to a point estimate for the part's requirement using the condition codes.

There are two major issues with this approach. First, it may find an infeasible point/range estimate (Appendix D.1 provides an example for this case). Second, it obtains a point estimate using an arbitrary demand realization while there are usually many consistent demand realizations (as alternative optimal solutions $a^*_{\mathbf{y}}$'s in step 2).

5.2.2. Comparison to current practice Fig. 5 provides a comparison between our approach and the current practice using a smaller instance that has 78 options' penetration rates. We consider different number of penetration rates denoted by n_1 (varies from 10 to 78) and focus on three parts denoted by $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$. For each n_1, we find a point estimate by applying the current approach described in subsection 5.2, and then, a range is created with 10% radius around the point estimate. "Practice" and "New" indicate the ranges found by applying the current approach and our method, respectively (the shaded area indicates the range found by our approach). We make the following observations:

- For small and medium n_1, the range found by their approach is usually a subset of the optimal range found by our approach which may result in shortage/excess of parts.
- Their approach sometimes (for medium and large n_1) finds a range that includes points outside of the optimal range that is found by our approach. In such cases, their approach suggests requirement that may never happen, i.e., they have excess capacity.
- For parts $\tilde{R}_1$, $\tilde{R}_2$, and $\tilde{R}_3$, the *mean absolute percentage errors* (MAPE) for the minimum values of the ranges obtained by their approach are respectively 30.95%, 17.13%, and 13.78%, and for the maximum values of those ranges are 8.76%, 25.04%, and 12.04%, respectively (compared to the minimum and maximum values of the optimal ranges found by our approach). On the average, the minimum and maximum values of the ranges found by their approach have MAPEs of 20.62% and 15.28%, respectively. Consequently, the average total error of the ranges, which is the sum of these two errors, is 35.9%.

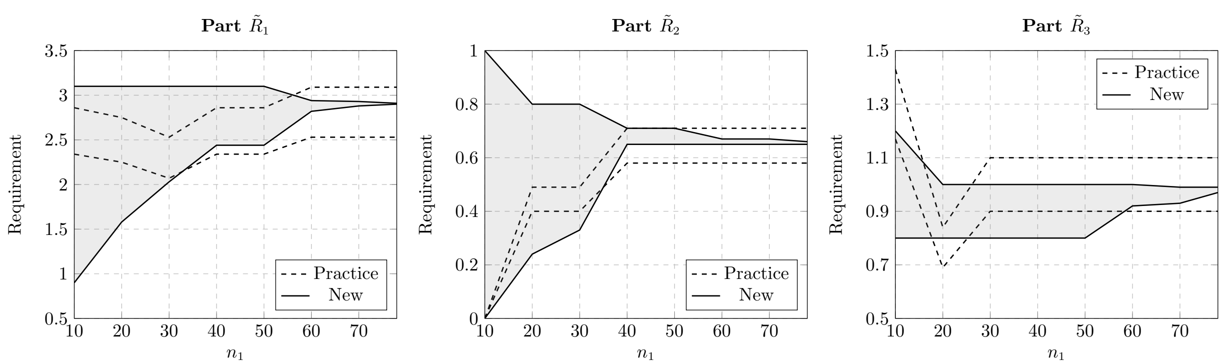

Figure 5 Our approach versus the current practice (the shaded areas have been determined by our approach).

5.3. Impact of additional information on the obtained ranges

Recall that the joint penetration rates of options can be incorporated as an additional penetration rate of a hypothetical option. Since we do not have the actual joint penetration rates, we use a subset of the given penetration rates as surrogate for additional information. For example, when we move from $n_1 = 10$ to $n_1 = 20$, the additional 10 penetration rates can be considered as additional information. Fig. 5 shows the effects of additional information on the obtained ranges by our method and the current approach.

- The width of the range found by our approach reduces as a result of incorporating new penetration rates, and for large n_1, the ranges are very tight for all parts. Whereas, in the current approach, since the radius of the range is always pre-specified, adding new information may not result in tightening the ranges. Furthermore, to obtain a satisfactory range using our approach, one could start with the given set of penetration rates and acquire new information as needed. For example, if 40 options initially have penetration rates, the range for parts $\tilde{R}_1$ and $\tilde{R}_2$ may be satisfactory while the range for part $\tilde{R}_3$ may be considered too wide. Hence, we may ask for more penetration rates to tighten the range for part $\tilde{R}_3$. Obviously, the cost for obtaining the additional penetration rates must be taken into consideration.
- In our approach, the minimum (respectively, maximum) value of the range is non-increasing (respectively, non-decreasing) in the information level n_1. This indicates that the obtained range for any information level n_1 always include the *actual value*—by the *actual value* we mean the requirement that correspond to the case of *full information* on the penetration rates of all options and joint penetration rates of options. However, the minimum and maximum values of the range produced by the current approach are not necessarily monotone and the obtained ranges may exclude the *actual value*. For example, for part $\tilde{R}_1$, when $n_1 \leq 50$ the range does not include the *actual value* that is found by our approach for large n_1. Similar error happens for parts $\tilde{R}_2$ and $\tilde{R}_3$ when $n_1 \leq 30$ and $n_1 \leq 20$, respectively.
- In the case of high information level (large n_1), our approach finds very tight ranges, and hence, we conclude that our approach is able to effectively absorb all available information and provide the tightest possible range for any information level n_1. In contrast, the range obtained by the current practice approach is unnecessarily wide for large n_1 indicating that the current approach is unable to utilize the given information and provides a range estimate based on a small subset of the penetration rates only.

In summary, the current approach may provide a range that excludes the *actual value* or a range that is unnecessarily wide. Whereas, our approach always utilizes all available information and finds the tightest possible range that includes the *actual value*.

6. Extensions

In this section, we extend our methodology in the following directions. First, we consider cases where options' penetration rates are given as points, ranges, or a combination of both. Second, based on the current practice in the GAM, usually a group of parts, with similar manufacturing requirements, are contracted to a single supplier. In these cases, the contract negotiation is also based on the total volume of the parts that are being subcontracted. Hence, in addition to providing ranges on individual parts' requirement, we provide a range estimate on the group of parts.

6.1. When ranges on options' penetration rates are given

Currently in the GAM, the forecast penetration rates of options are single values denoted by $\hat{p}_i$'s. What if options' penetration rates are given as ranges for some/all options or a combination of points for some options and ranges for some other options? In this subsection, we present a general approach that incorporates such scenarios. Let $p_{L,i}$ and $p_{U,i}$ denote the given lower and upper bounds on the penetration rate of option i satisfying $0 \le p_{L,i} \le p_{U,i} \le 1$. Thus, if an option has a single value forecast, then, $p_{L,i} = p_{U,i}$, and if an option has not been assigned any forecast, then, $p_{L,i} = 0$ and $p_{U,i} = 1$. This generalization of problem (P1) is formulated as:

$$\text{(PG):} \quad \min / \max \quad Q_R = \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \tag{5}$$

$$\text{s.t.} \quad p_{L,i} \le p_i \le p_{U,i}, \ \forall i \in \mathcal{N}, \tag{6}$$

$$\mathbf{p} \in conv(\mathbb{Y}). \tag{7}$$

Problem (PG) can be formulated as a mixed-integer linear program similar to Appendix A.1. To solve industrial instances of (PG), based on a similar motivation that we discussed in section 3, we present an equivalent nonlinear program:

$$\text{(PG}'\text{):} \quad \min / \max \quad \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \pm M \sum_{i=1}^{n} \max\{p_{L,i} - p_i, 0, p_i - p_{U,i}\} \tag{8}$$

$$\text{s.t.} \quad \mathbf{p} \in conv(\mathbb{Y}), \tag{9}$$

where, for sufficiently large M, solving (PG$'$) provides an optimal solution to (PG). Theorem 2 states this equivalence.

THEOREM 2 **(Generalization of Nonlinear Programming Equivalence)**. *An optimal solution of problem (PG$'$) is an optimal solution to problem (PG) if*

$$M > \left(\sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha \right) \left(1 + \frac{2n+1}{\sqrt{\lambda_{\min}(n)}} \right),$$

where $\lambda_{\min}(n)$ denotes the smallest eigenvalue of all matrices in the form of $B^\top B$ where B is an invertible matrix of size $(2n+1)\times(2n+1)$ with all entries being a member of $\{-1,0,1\}$.

The proof is given in Appendix E.1. This theorem simply means that for sufficiently large M, one can solve (PG′) to obtain an optimal solution to (PG). In addition, it specifies a lower bound on M as a function of α's, n, and $\lambda_{\min}(n)$. An increase in the values of α's and/or n require using larger values for M. Note that $\lambda_{\min}(n)$ is itself a function of n and it exists for a fixed n (because there is a finite number of possibilities for matrix B and the smallest eigenvalue for each possibility can be computed). Wolkowicz and Styan (1980) and Merikoski and Virtanen (1997) present lower bounds on the smallest eigenvalue of a symmetric positive definite matrix.

We remark that the minimization (respectively, maximization) case of problem (PG′) has a piecewise linear convex (respectively, concave) objective function, and hence, its subgradient vector can be determined at any given point. In addition, the feasible set of (PG′) is similar to that of (P2) and therefore a similar algorithm presented in subsection 3.2 can be applied.

6.2. Obtaining a range for a group of parts

We characterize the requirement of a *group of parts* that require a similar manufacturing processes and are contracted to the same supplier—i.e., $\sum_{R\in\tilde{\mathcal{R}}} Q_R$, for some $\tilde{\mathcal{R}} \subseteq \mathcal{R}$. Clearly, if we have unique values for Q_R's (not ranges), we can then determine the sum by summing up those values. However, finding a range for $\sum_{R\in\tilde{\mathcal{R}}} Q_R$ is complicated. Our approach is based on creating a hypothetical part which incorporates all condition codes for the parts that are included in the group. By finding a range on the requirement of this hypothetical part, we obtain a range on the requirement of the *group of parts*. The following proposition presents our approach.

Proposition 4 **(Requirement for a Group of Parts)**. *Let* $\tilde{\mathcal{R}} \subseteq \mathcal{R}$, $\tilde{\mathcal{R}} \neq \{\}$. *Then,* $\sum_{R\in\tilde{\mathcal{R}}} Q_R = Q_{\hat{R}}$, *where* $\hat{R}$ *is a hypothetical part with the following set of condition codes:*

$$\mathscr{C}(\hat{R}) = \{(F,\alpha')|F \in \bigcup_{R\in\tilde{\mathcal{R}}} \mathscr{C}\mathscr{F}(R), \alpha' = \sum_{R\in\tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)} \alpha\},$$

and $\mathscr{C}\mathscr{F}(R) := \{F|\exists(F,\alpha) \in \mathscr{C}(R)\}$.

The proof is given in Appendix E.2. Therefore, using proposition 4, the requirement of the group of parts is given by $Q_{\hat{R}} = \sum_{(F,\alpha)\in\mathscr{C}(\hat{R})} \alpha p(F)$. Appendix E.3 provides an illustration of Proposition 4.

Next, we extend our result to obtain a range on the requirement of a *resource* that is used by the supplier (not the manufacturer) to produce a group of parts. A *resource* consumption can be represented as a nonnegative combination of the parts' requirement, where the coefficients indicate the unit *resource* usage for producing parts. Let $\tilde{\mathcal{R}}_{\mathrm{X}} \subseteq \mathcal{R}$ denote the subset (group) of parts that require resource X. We focus on a single resource and our analysis must be repeated for each resource. For ease of notation, in the remainder, we drop index X and refer to resource X simply as "the resource." The following proposition presents our approach.

PROPOSITION 5 **(Consumption Rate of a Resource)**. *Let $\tilde{\mathcal{R}} \subseteq \mathcal{R}$, $\tilde{\mathcal{R}} \neq \{\}$, and $\eta_R \in \mathbb{R}_{++}$, for all $R \in \tilde{\mathcal{R}}$, be given. Then, $\sum_{R\in\tilde{\mathcal{R}}} \eta_R Q_R = Q_{\hat{R}}$, where $\hat{R}$ is a hypothetical part with the following set of condition codes:*

$$\mathscr{C}(\hat{R}) = \{(F,\alpha')|F \in \bigcup_{R\in\tilde{\mathcal{R}}} \mathscr{C}\mathscr{F}(R), \alpha' = \sum_{R\in\tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha)\in\mathscr{C}(R)} \alpha\eta_R\},$$

and $\mathscr{C}\mathscr{F}(R) := \{F|\exists(F,\alpha) \in \mathscr{C}(R)\}$.

The proof is similar to Proposition 4 and hence is skipped. Thus, the resource consumption rate is given by $Q_{\hat{R}} = \sum_{(F,\alpha)\in\mathscr{C}(\hat{R})} \alpha p(F)$. We finally note that our approach generates ranges for all resources that are used to produce a group of parts, which is useful for both the supplier and manufacturer in contract negotiations.

7. Conclusions

We study a parts-capacity planning problem in the context of mass customization. We develop a new methodology for finding the ranges on the parts' requirement when the demand forecast on configurations does not exist and the manufacturers forecast a demand on options. We present a formulation that enables us to generate a range on the parts' requirement and propose an effective methodology to solve it. We show how additional information on the joint options' penetration rates impacts the parts' requirement ranges and design an interactive method for achieving a desired target range. We apply our method to an industrial instance provided by the GAM and compare our results to the current approach. We finally extend our methodology to the following cases: (i) when ranges on the options' penetration rates are given, and (ii) finding a range on the requirement of a group of parts.

Our algorithm for solving problem (P2) has the following shortcoming. First, since the objective function is not smooth, the algorithm may converge slowly and lacks a convergence rate guarantee. However, since we have a bound on the optimal range, we can assess the optimality gap when the algorithm is terminated prior to completion. Further research on improving the speed of our algorithm, obtaining a convergence rate guarantee, and designing more effective bounds would be helpful.

Finally, we propose the following directions for future research. First, in parts' capacity management, range on a part's requirement provides one dimension of the variability. Another dimension of variability is the standard deviation which can be used to truncate the ranges for parts' requirements. Developing such measures would support parts' capacity management. Second, in this paper, we consider an strategic parts' capacity planning that is performed prior to launching the product. This is an static model that determines a range for each part's requirement. Once the product is launched, parts' requirement can dynamically change based on demand evolution. The dynamic parts' capacity planning is a critical problem to be considered.

Endnotes

1. Note that, in a CPS, rates add up to 1; in an OPS, penetration rates are between 0 and 1 but they may not add up to 1; and, a part's requirement may be greater than 1, e.g. each car has 4 tiers.
2. Formulating logical propositions as linear constraints is also discussed in Yan and Hooker (1999) and Chandru and Hooker (1999).
3. We note that the number of sub-configurations is exponential in the size of OP_R and hence a limited number of additional options can be incorporated.
4. We note that, in real instances, some options exist without forecast penetration rates—options that are not customer facing but are options available for internal operations.

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Appendix A: Supporting Material for Section 2

A.1. Mixed-integer linear programming formulation

Note that $conv(\mathbb{Y})$ is bounded by a unit hypercube in $\mathbb{R}^n$. Moreover, a point inside a polytope (bounded polyhedron) in $\mathbb{R}^n$ can be represented as a convex combination of at most $n+1$ extreme points (Bertsimas and Tsitsiklis 1997). Hence, if $\mathbf{p} \in conv(\mathbb{Y})$, then $\mathbf{p}$ can be represented as a convex combination of at most $n+1$ extreme points of $conv(\mathbb{Y})$. Therefore, there exist $\mathbf{y}^1, \mathbf{y}^2, \ldots, \mathbf{y}^n, \mathbf{y}^{n+1} \in \mathbb{Y}$ and $\mathbf{a} = (a_1, \ldots, a_n, a_{n+1}) \in \mathbb{R}_+^{n+1}$ such that $\mathbf{p} = \sum_{j=1}^{n+1} a_j \mathbf{y}^j$ and $\sum_{j=1}^{n+1} a_j = 1$. Thus, $\mathbf{p} \in conv(\mathbb{Y})$ can be formulated as the problem of finding, at most, $n+1$ producible configurations such that $\mathbf{p}$ belongs to their convex hull. Thus, we formulate our MILP problem as follows:

$$\text{(P1):} \quad \min/\max \ Q_R = \sum_{(i,\alpha)\in\widehat{\mathscr{C}}(R)} \alpha p_i \tag{10}$$

$$\text{s.t.} \quad p_i = \hat{p}_i, \ \forall i \in \mathcal{N}_1, \tag{11}$$

$$z_i^j \le y_i^j, \ \forall i, j, \tag{12}$$

$$y_i^j - 1 \le z_i^j - a_j \le 1 - y_i^j, \ \forall i, j \tag{13}$$

$$p_i = \sum_{j=1}^{n+1} z_i^j, \ \forall i, \tag{14}$$

$$\sum_{j=1}^{n+1} a_j = 1, \tag{15}$$

$$a_j \ge 0, \ z_i^j \ge 0, \ \mathbf{y}^j \in \mathbb{Y}, \ \forall i, j. \tag{16}$$

Eqs. (12)-(14) are equivalent to the constraints $p_i = \sum_{j=1}^{n+1} a_j y_i^j$, for all $i = 1, \ldots, n$. Note that these constraints are not linear because of the multiplication of variables a_j and y_i^j. Eqs. (12)-(14) provide an approach to formulating these constraints using linear (in)equalities. We discuss this in more detail. For all i, j, we define new variables $z_i^j \ge 0$ such that $z_i^j = a_j y_i^j$. If $y_i^j = 0$, then $z_i^j = 0$, and if $y_i^j = 1$, then $z_i^j = a_j$. This is guaranteed by Eqs. (12)-(14). Eq. (15) ensures that the coefficients of the producible configurations in the convex combination add up to 1.

Fig. 6 shows the solution of our MILP problem for part R_1 in Example 2. We obtain the range $2.155 \le Q_{R_1} \le 2.585$ and the width of this range is 0.43 (20% of the minimum value). Note that some of the $\mathbf{y}^j$'s are equal; e.g., $\mathbf{y}^8 = \mathbf{y}^{11}$ in the minimization problem. This means that the optimal solution can be represented as a convex combination of less than $n+1$ extreme points of $conv(\mathbb{Y})$.

Appendix B: Supporting Material for Section 3

B.1. Proof of Theorem 1: Nonlinear Programming Equivalence

We need the following definitions in this proof. Let $\mathcal{N}_3$ denote the set of *artificial* options defined to simplify the complex condition codes for part R, and define $n_3 := |\mathcal{N}_3|$. Moreover, define $\mathcal{N}_2 :=$

Minimization problem:
Optimal value: 2.155
$\mathbf{p} = (0.385, 0.904, 0.019, 0.861, 0.096, 0.019, 0.05, 0.046, 0.289, 0.385, 0.885, 0)$
$\mathbf{a} = (0, 0.024, 0.572, 0, 0.019, 0, 0.027, 0.019, 0.05, 0.289, 0, 0, 0)$
$\mathbf{y}^1 = (1,0,0,1,1,1,0,1,0,1,0,0)$
$\mathbf{y}^2 = (0,1,0,0,0,0,0,0,0,0,1,0)$
$\mathbf{y}^3 = (0,1,0,1,0,0,0,0,0,0,1,0)$
$\mathbf{y}^4 = (0,1,0,0,0,0,0,0,0,0,1,0)$
$\mathbf{y}^5 = (0,1,1,0,0,0,0,0,0,0,0,0)$
$\mathbf{y}^6 = (0,1,1,0,1,0,1,0,0,0,0,0)$
$\mathbf{y}^7 = (1,0,0,0,1,0,0,1,0,1,0,0)$
$\mathbf{y}^8 = (1,0,0,0,1,1,0,1,0,1,0,0)$
$\mathbf{y}^9 = (1,0,0,0,1,0,1,0,0,1,0,0)$
$\mathbf{y}^{10} = (1,1,0,1,0,0,0,0,1,1,1,0)$
$\mathbf{y}^{11} = (1,0,0,0,1,1,0,1,0,1,0,0)$
$\mathbf{y}^{12} = (1,0,0,1,1,1,0,1,0,1,0,0)$
$\mathbf{y}^{13} = (1,0,0,1,1,1,0,1,0,1,0,0)$.

Maximization problem:
Optimal value: 2.585
$\mathbf{p} = (0.385, 1, 0.019, 0.861, 0.019, 0.019, 0.05, 0.046, 0.385, 0.385, 1, 0.05)$
$\mathbf{a} = (0, 0, 0.05, 0, 0.227, 0.062, 0.027, 0, 0, 0, 0.615, 0.019, 0)$
$\mathbf{y}^1 = (1,1,0,1,0,0,1,0,1,1,1,1)$
$\mathbf{y}^2 = (1,1,1,1,0,0,1,1,1,1,1,1)$
$\mathbf{y}^3 = (1,1,0,0,0,0,1,0,1,1,1,1)$
$\mathbf{y}^4 = (1,1,0,0,0,0,1,0,1,1,1,1)$
$\mathbf{y}^5 = (1,1,0,1,0,0,0,0,1,1,1,0)$
$\mathbf{y}^6 = (1,1,0,0,0,0,0,0,1,1,1,0)$
$\mathbf{y}^7 = (1,1,0,0,0,0,0,1,1,1,1,0)$
$\mathbf{y}^8 = (1,0,0,0,1,1,1,0,0,1,0,0)$
$\mathbf{y}^9 = (1,0,0,1,1,1,0,1,0,1,0,0)$
$\mathbf{y}^{10} = (1,0,1,0,1,0,1,1,1,1,0,0)$
$\mathbf{y}^{11} = (0,1,0,1,0,0,0,0,0,0,1,0)$
$\mathbf{y}^{12} = (1,1,1,1,1,1,0,1,1,1,1,0)$
$\mathbf{y}^{13} = (1,1,1,1,1,1,0,1,1,1,1,0)$.

Hence, the obtained range for Q_{R_1} is:
$2.155 \leq Q_{R_1} \leq 2.585$.
Similarly, for parts R_2 and R_3, we find:
$0 \leq Q_{R_2} \leq 0.019$
$0.765 \leq Q_{R_3} \leq 0.918$.

Figure 6 The solution of our MILP problem for parts R_1, R_2, and R_3 (also see Example 2, Fig. 2).

$\mathcal{N} \backslash \{\mathcal{N}_1 \cup \mathcal{N}_3\}$, and $n_2 := |\mathcal{N}_2|$. Define $\mathscr{C}_1(R) := \{(i,\alpha) \in \mathscr{C}(R) | i \in \mathcal{N}_1\}$, $\mathscr{C}_2(R) := \{(i,\alpha) \in \mathscr{C}(R) | i \in \mathcal{N}_2\}$, and $\mathscr{C}_3(R) := \{(i,\alpha) | i \in \mathcal{N}_3, \exists (F,\alpha) \in \mathscr{C}(R) : F \Leftrightarrow i\}$. Hence, $\tilde{\mathscr{C}}(R) = \mathscr{C}_1(R) \cup \mathscr{C}_2(R) \cup \mathscr{C}_3(R)$.

In the remainder of this proof, we sometimes drop index R for the ease of notation. Define vector $\beta \in \mathbb{R}^n$ as follows: for all $i = 1, \ldots, n$, let $\beta_i := \sum_{(i,\alpha) \in \tilde{\mathscr{C}}(R)} \alpha$. Moreover, let $\mathbf{p}_L^*$ and $\mathbf{p}_U^*$ denote the optimal solutions of (P2L) and (P2U), respectively.

Recall that the objective function of problem (P1) is $Q_R = \sum_{(i,\alpha) \in \tilde{\mathscr{C}}(R)} \alpha p_i$. Since $\tilde{\mathscr{C}}(R) = \bigcup_{\ell=1}^3 \mathscr{C}_\ell(R)$ and $\mathscr{C}_\ell(R)$'s are mutually exclusive, then the objective function of problem (P1) can be written as $Q_R = \sum_{\ell=1}^3 \sum_{(i,\alpha) \in \mathscr{C}_\ell(R)} \alpha p_i$. Thus, problem (P1) is equivalent to:

$$\min / \max \sum_{\ell=1}^{3} \sum_{(i,\alpha) \in \mathscr{C}_\ell(R)} \alpha p_i \tag{17}$$

$$\text{s.t.} \quad p_i = \hat{p}_i, \quad \forall i \in \mathcal{N}_1, \; \mathbf{p} \in conv(\mathbb{Y}). \tag{18}$$

We propose a relaxation of this problem by relaxing the equality constraints $p_i = \hat{p}_i$, for all $i \in \mathcal{N}_1$, and adding (respectively, subtracting) $\sum_{i \in \mathcal{N}_1} M|p_i - \hat{p}_i|$ to the objective function of (P1min) (respectively, (P1max)). This results in the following problems:

$$\text{(P1Rmin):} \quad \min_{\mathbf{p} \in conv(\mathbb{Y})} \Big\{ \sum_{\ell=1}^{3} \Big\{ \sum_{(i,\alpha) \in \mathscr{C}_\ell(R)} \alpha p_i \Big\} + \sum_{i \in \mathcal{N}_1} M|p_i - \hat{p}_i| \Big\},$$

$$\text{(P1Rmax):} \quad \max_{\mathbf{p} \in conv(\mathbb{Y})} \Big\{ \sum_{\ell=1}^{3} \Big\{ \sum_{(i,\alpha) \in \mathscr{C}_\ell(R)} \alpha p_i \Big\} - \sum_{i \in \mathcal{N}_1} M|p_i - \hat{p}_i| \Big\}.$$

By (P1R), we refer to both problems (P1Rmin) and (P1Rmax). In the following lemma, we show that, if M is sufficiently big, then the optimal solution of the relaxation problem (P1R) is also an optimal solution for (P1).

LEMMA 3. *For a fixed β, there exists M such that solving problem (P1R) provides an optimal solution for (P1).*

Proof. Note that this lemma is an special case of Theorem 2 that we will state and prove in subsection 6.1 and Appendix E.1. Theorem 2 proves the existence of M for the case where the *Marketing Constraints* are replaced with:

$$p_{L,i} \le p_i \le p_{U,i}, \quad \forall i \in \mathcal{N},$$

where $p_{L,i}$ and $p_{U,i}$ are the forecast lower and upper bounds on the penetration rate of option i. In other words, the forecast penetration rate of option i is given as a range $[p_{L,i}, p_{U,i}]$ rather than a single point. On the other hand, recall that the *Marketing Constraints* in (P1) are:

$$p_i = \hat{p}_i, \quad \forall i \in \mathcal{N}_1.$$

Since $\mathcal{N}_1 \subseteq \mathcal{N}$, and each penetration rate is always between 0 and 1, then the *Marketing Constraints* in (P1) can be equivalently written as:

$$\hat{p}_i \le p_i \le \hat{p}_i, \quad \forall i \in \mathcal{N}_1,$$
$$0 \le p_i \le 1, \quad \forall i \in \mathcal{N} \backslash \mathcal{N}_1.$$

Hence, by applying Theorem 2 the proof is complete. □

Recall that $Q^*_{R,L}$ and $Q^*_{R,U}$ denote the optimal values of problems (P1min) and (P1max), respectively. We first prove the theorem for problem (P1min) by showing $Q^*_{R,L} = \beta^T \mathbf{p}^*_L$.

Using the equivalence between (P1min) and (P1Rmin), we have:

$$Q^*_{R,L} = \sum_{(i,\alpha)\in\mathscr{C}_1(R)} \alpha\hat{p}_i + \min_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} + \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\}. \tag{19}$$

Note that, based on the definition of β, we have: $\sum_{(i,\alpha)\in\mathscr{C}_1(R)} \alpha\hat{p}_i = \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i$. Moreover, we have: $\sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i = \sum_{i\in\mathcal{N}_\ell} \beta_i p_i$, for all $\ell = 2,\ 3$. Hence, Eq. (19) can be written as:

$$Q^*_{R,L} = \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i + \min_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \Big\{ \sum_{i\in\mathcal{N}_\ell} \beta_i p_i \Big\} + \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\}. \tag{20}$$

Since $p_i \ge 0$, for all $i \in \mathcal{N}$, we have: $\sum_{i\in\mathcal{N}_\ell} \beta_i p_i = \sum_{i\in\mathcal{N}_\ell} \beta_i |p_i - 0|$, for all $\ell = 2,\ 3$. Therefore, using the definition of $\mathbf{w}$ and $\hat{\mathbf{p}}_L$, the objective function of the minimization problem in Eq. (20) is equal to $\sum_{i\in\mathcal{N}} w_i |p_i - \hat{p}_{L,i}|$. Thus,

$$Q^*_{R,L} = \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i + \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_L)\|_1. \tag{21}$$

Define $\mathbf{p}^*_L := \arg\min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_L)\|_1$. Using Lemma 3, we have $p^*_{L,i} = \hat{p}_i$, for all $i \in \mathcal{N}_1$. Therefore, using Eq. (20), we have:

$$Q^*_{R,L} = \sum_{i\in\mathcal{N}_1} \beta_i p^*_{L,i} + \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i p^*_{L,i} = \beta^T \mathbf{p}^*_L,$$

and hence, the proof of $Q^*_{R,L} = \beta^T \mathbf{p}^*_L$ is complete. The proof for $Q^*_{R,U} = \beta^T \mathbf{p}^*_U$ has similar steps that we summarize as follows:

$$\begin{aligned}
Q^*_{R,U} &= \max_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=1}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} - \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{(i,\alpha)\in\mathscr{C}_1(R)} \alpha\hat{p}_i + \max_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \Big\{ \sum_{(i,\alpha)\in\mathscr{C}_\ell(R)} \alpha p_i \Big\} - \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i + \max_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i p_i - \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i + \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i - \min_{\mathbf{p}\in conv(\mathbb{Y})} \Big\{ \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i(1-p_i) + \sum_{i\in\mathcal{N}_1} M|p_i - \hat{p}_i| \Big\} \\
&= \sum_{i\in\mathcal{N}_1} \beta_i\hat{p}_i + \sum_{\ell=2}^{3} \sum_{i\in\mathcal{N}_\ell} \beta_i - \min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_U)\|_1 \\
&= \beta^T \mathbf{p}^*_{\mathbf{U}},
\end{aligned}$$

where, $\mathbf{p}^*_{\mathbf{U}} := \arg\min_{\mathbf{p}\in conv(\mathbb{Y})} \|\mathbf{Diag}(\mathbf{w})(\mathbf{p} - \hat{\mathbf{p}}_U)\|_1$. To obtain the fourth line, we add and subtract $\sum_{\ell=2}^{3}\sum_{i\in\mathcal{N}_\ell} \beta_i$ to the objective function of the maximization problem in the third line. Then, since $+\sum_{\ell=2}^{3}\sum_{i\in\mathcal{N}_\ell} \beta_i$ is constant, we bring it outside of the maximization problem. Moreover, we multiply the objective function by -1, which results in a minimization problem. To obtain the fifth line, we note that $1 - p_i \geq 0$, for all i; hence, $1 - p_i = |p_i - 1|$, for all i. We then apply the definition of $\hat{\mathbf{p}}_U$. The last line follows by applying the optimal solution $\mathbf{p}^*_{\mathbf{U}}$ to the third line and noting that $\sum_{i\in\mathcal{N}_1} M|p^*_{U,i} - \hat{p}_i| = 0$ (because of Lemma 3).

B.2. Proof of Proposition 2: Bound on the Optimal Range

Recall that problem (P1min) (respectively, (P1max)) finds a point in $conv(\mathbb{Y})$ that satisfies the marketing constrains and minimizes (respectively, maximizes) the requirement of part R. Recall that $a_{\mathbf{y}}$ denotes the coefficient of configuration $\mathbf{y} \in \mathbb{Y}$ in a CPS. Moreover, with some abuse of notation, let $\varsigma_{\mathbf{y}}$ denote the number of units of part R required to produce one unit of configuration $\mathbf{y} \in \mathbb{Y}$. In fact, problem (P1) is equivalent to:

$$\begin{aligned}
(\text{P1}'):\quad \min/\max\ \ & Q_R = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}}\varsigma_{\mathbf{y}} \\
\text{s.t.}\ \ & \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} y_i = \hat{p}_i,\ \ \forall i\in\mathcal{N}_1, \\
& \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} = 1, \\
& a_{\mathbf{y}} \geq 0,\quad \forall \mathbf{y}\in\mathbb{Y}.
\end{aligned}$$

Let us reorder the options such that $\mathbf{y} \in \mathbb{Y}$ can be written in the form of $\mathbf{y} = \binom{\mathbf{y}_1}{\mathbf{y}_2}$ where $\mathbf{y}_1$ contains the values of options in OP_R and $\mathbf{y}_2$ contains the values of options not in OP_R. Observe that $\varsigma_{\mathbf{y}_1} = \varsigma_{\mathbf{y}}$, for all $\mathbf{y} = \binom{\mathbf{y}_1}{\mathbf{y}_2} \in \mathbb{Y}$. Thus, the objective function of (P1′) can be written as:

$$Q_R = \sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}}\varsigma_{\mathbf{y}} = \sum_{\mathbf{y}_1\in \text{SC}_R} \varsigma_{\mathbf{y}_1} \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}}.$$

Similarly, the marketing constraints can be written as:

$$\sum_{\mathbf{y}_1\in \text{SC}_R} y_i \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} = \hat{p}_i,\ \forall i \in \mathcal{N}_1 \cap \text{OP}_R, \tag{22}$$

$$\sum_{\mathbf{y}\in\mathbb{Y}} a_{\mathbf{y}} y_i = \hat{p}_i,\ \forall i \in \mathcal{N}_1 \backslash \text{OP}_R. \tag{23}$$

Consider a relaxation of (P1′) by dropping constraints (23):

$$\begin{aligned}
(\text{P1}''):\quad \min/\max\ \ Q''_R = & \sum_{\mathbf{y}_1\in \text{SC}_R} \varsigma_{\mathbf{y}_1} \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} \\
\text{s.t.}\ & \sum_{\mathbf{y}_1\in \text{SC}_R} y_i \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} = \hat{p}_i,\ \forall i \in \mathcal{N}_1 \cap \text{OP}_R, \\
& \sum_{\mathbf{y}_1\in \text{SC}_R} \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}} = 1, \\
& a_{\mathbf{y}} \geq 0, \quad \forall \mathbf{y} \in \mathbb{Y}.
\end{aligned}$$

Let $Q''_{R,L}$ and $Q''_{R,U}$ denote the optimal values of the minimization and maximization cases of problem (P1″), respectively. We next show that given a feasible solution of (P1″), there exists a feasible solution of (PB) with the same objective value. Consider a feasible solution $a_{\mathbf{y}}$'s of problem (P1″). Define $\bar{a}_{\mathbf{y}_1} := \sum_{\mathbf{y}=\binom{\mathbf{y}_1}{\mathbf{y}_2}\in\mathbb{Y}} a_{\mathbf{y}}$, for all $\mathbf{y}_1 \in \text{SC}_R$. Note that $\bar{a}_{\mathbf{y}_1}$'s satisfy all constraints of (PB) and the corresponding objective value is equal to that of $a_{\mathbf{y}}$'s in problem (P1″). This implies that: $Q^{\circ}_{R,L} \leq Q''_{R,L}$ and $Q''_{R,U} \leq Q^{\circ}_{R,U}$. In addition, since (P1″) is a relaxation of (P1), we have: $Q''_{R,L} \leq Q^*_{R,L}$ and $Q^*_{R,U} \leq Q''_{R,U}$. By combining these inequalities, we obtain that the optimal values of problems (P1) and (PB) satisfy: $Q^{\circ}_{R,L} \leq Q^*_{R,L}$ and $Q^*_{R,U} \leq Q^{\circ}_{R,U}$. Hence, the proof is complete.

Appendix C: Supporting Material for Section 4

C.1. Proof of Lemma 1: Consistency of a New Information Set

Consider adding information $p(F) = \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$. This is equivalent to adding $p_{A'} = \sum_{\mathbf{y}\in\mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F)$ and $A' \Leftrightarrow F$, where A' is an artificial option. Note that the updated set of producible configurations after adding option A' is as follows:

$$\mathbb{Y}' := \left\{ \begin{pmatrix} \mathbf{y} \\ y_{A'} \end{pmatrix} \middle| \mathbf{y} \in \mathbb{Y}, y_{A'} = v_{\mathbf{y}}(F) \right\}.$$

Moreover, define $\tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}} := a_{L,\mathbf{y}}$, for all $\binom{\mathbf{y}}{y_{A'}} \in \mathbb{Y}'$. Note that $\tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}}$'s are feasible coefficients for the updated producible configurations in the new problem. Hence, we only need to show that:

$$p_{A'} = \sum_{\binom{\mathbf{y}}{y_{A'}} \in \mathbb{Y}'} \tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}} y_{A'} = \sum_{\mathbf{y} \in \mathbb{Y}} a_{L,\mathbf{y}} v_{\mathbf{y}}(F).$$

Note that this equation holds because $\tilde{a}_{L,\binom{\mathbf{y}}{y_{A'}}} := a_{L,\mathbf{y}}$, for all $\binom{\mathbf{y}}{y_{A'}} \in \mathbb{Y}'$, and $y_{A'} = v_{\mathbf{y}}(F)$.

Similarly, we can show that adding information $p(F) = \sum_{\mathbf{y} \in \mathbb{Y}} a_{U,\mathbf{y}} v_{\mathbf{y}}(F)$ is also feasible. Finally, since $p(F) = \gamma_{L,F}$ and $p(F) = \gamma_{U,F}$ are consistent, then $p(F) = \gamma$ is consistent for all γ such that $\min\{\gamma_{L,F}, \gamma_{U,F}\} \le \gamma \le \max\{\gamma_{L,F}, \gamma_{U,F}\}$.

C.2. Proof of Proposition 3: Ordering Different Information Sets

Define A' as a new artificial option, and consider the following three problems.

$$\begin{aligned}
\text{Problem (i):}\ \min/\max \quad & \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \\
\text{s.t.}\quad & p_i = \hat{p}_i,\quad \forall i \in \mathcal{N}_1,\ \mathbf{p} \in conv(\mathbb{Y}),\\
& p_{A'} = \gamma,\ A' \Leftrightarrow F',\\
\text{Problem (ii):}\ \min/\max \quad & \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \\
\text{s.t.}\quad & p_i = \hat{p}_i,\quad \forall i \in \mathcal{N}_1,\ \mathbf{p} \in conv(\mathbb{Y}),\\
& p_{A'} = \gamma,\ A' \Leftrightarrow F' \wedge F'',\\
\text{Problem (iii):}\ \min/\max \quad & \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha p_i \\
\text{s.t.}\quad & p_i = \hat{p}_i,\quad \forall i \in \mathcal{N}_1,\ \mathbf{p} \in conv(\mathbb{Y}),\\
& p_{A'} = \gamma,\ A' \Leftrightarrow F' \vee F''.
\end{aligned}$$

In all problems, there are $n+1$ binary variables that can be shown as a (n+1)-dimensional binary vector $\tilde{\mathbf{y}} = \binom{\mathbf{y}}{y_{A'}}$. Moreover, let $\tilde{\mathbf{p}} = \binom{\mathbf{p}}{p_{A'}}$. Finally, note that a feasible $\tilde{\mathbf{p}}$ can be written as a convex combination of some producible configurations. Consider the following cases:

<u>(a) $\gamma = 0$:</u> In this case $p_{A'} = 0$, meaning that in all problems, we must have $y_{A'} = 0$ for all producible configurations that have positive coefficients in the convex combination. In other words, if $\tilde{\mathbf{p}} = \binom{\mathbf{p}}{0}$, then $\tilde{\mathbf{p}}$ can be written as a convex combination of some producible configurations in the form of $\binom{\mathbf{y}}{0}$. Hence, we can restrict ourselves to $\tilde{\mathbf{y}}$'s with the last entry 0. In this case, in problem (i), we must have $(v(F'), v(F'')) \in \{(0,0),(0,1)\}$; in problem (ii), we must have $(v(F'), v(F'')) \in \{(0,0),(0,1),(1,0)\}$; and in problem (iii), we must have $(v(F'), v(F'')) \in \{(0,0)\}$. Thus, any $\binom{\mathbf{y}}{0}$ that is feasible for (iii) is also feasible for (i), and any $\binom{\mathbf{y}}{0}$ that is feasible for (i) is also feasible for

(ii). Thus, in this case, the optimal solution of (iii) is feasible for (i) and the optimal solution of (i) is feasible for (ii), meaning that:

$$Q^{*''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*'''}_{R,L} \leq Q_R \leq Q^{*'''}_{R,U} \leq Q^{*'}_{R,U} \leq Q^{*''}_{R,U}.$$

(b) $\gamma = 1$: In this case we can restrict ourselves to $\tilde{\mathbf{y}}$'s with the last entry equal to 1. Hence, in problem (i), we must have $(v(F'), v(F'')) \in \{(1,0),(1,1)\}$; in problem (ii), we must have $(v(F'), v(F'')) \in \{(1,1)\}$; and in problem (iii), we must have $(v(F'), v(F'')) \in \{(0,1),(1,0),(1,1)\}$. Thus, any $\binom{\mathbf{y}}{1}$ that is feasible for (ii) is also feasible for (i), and any $\binom{\mathbf{y}}{1}$ that is feasible for (i) is also feasible for (iii). Thus, in this case, the optimal solution of (ii) is feasible for (i), and the optimal solution of (i) is feasible for (iii), meaning that:

$$Q^{*'''}_{R,L} \leq Q^{*'}_{R,L} \leq Q^{*''}_{R,L} \leq Q_R \leq Q^{*''}_{R,U} \leq Q^{*'}_{R,U} \leq Q^{*'''}_{R,U}.$$

C.3. An interactive approach for reducing a part's requirement range

Fig. 7 shows the flowchart of our proposed interactive method. The flowchart consists of two components: the expert's decisions and the system's decisions. The "expert" provides new information, and the "system" solves PCPP and performs some systematic operations. First, the expert determines a subset of formulas $\tilde{\mathcal{F}} \subseteq \mathcal{F}$ for which new information can be found. The set $\tilde{\mathcal{F}}$ is usually large but enumerable. For example, $\tilde{\mathcal{F}}$ can be the set of all conjunctions of one or two positive options.

Second, the system solves PCPP and checks whether the obtained range is acceptable. This decision is made by a pre-defined criteria. For example, a common criteria is that the width of the range should not be bigger than 20% of the minimum value.

Third, if the range is not acceptable, the system calculates $|\gamma_{U,F} - \gamma_{L,F}|$ for all $F \in \tilde{\mathcal{F}}$, and creates a short list by choosing some of the formulas with the biggest nonzero values for $|\gamma_{U,F} - \gamma_{L,F}|$. The number of formulas in the short list is usually pre-defined—e.g., the short list consists of 10 formulas at most.

Last, the expert chooses a formula from the short list and determines its penetration rate. The system adds this new information set to it and solves the problem again. This process is repeated until an acceptable range is found.

Fig 8 illustrates our interactive method on part R_1 that we introduced in Example 2. We assume that the obtained range is acceptable if the width of the range is less than 10% of the minimum value. The short list (in steps (4) and (8)) consists of 4-5 formulas, with the allowable interval for the formulas' penetration rates. For example, in step (4), $0.246 \leq p(A_1 \wedge A_4) \leq 0.347$ means that the expert can assign a value to the penetration rate of formula $A_1 \wedge A_4$ in the interval $[0.246, 0.347]$.

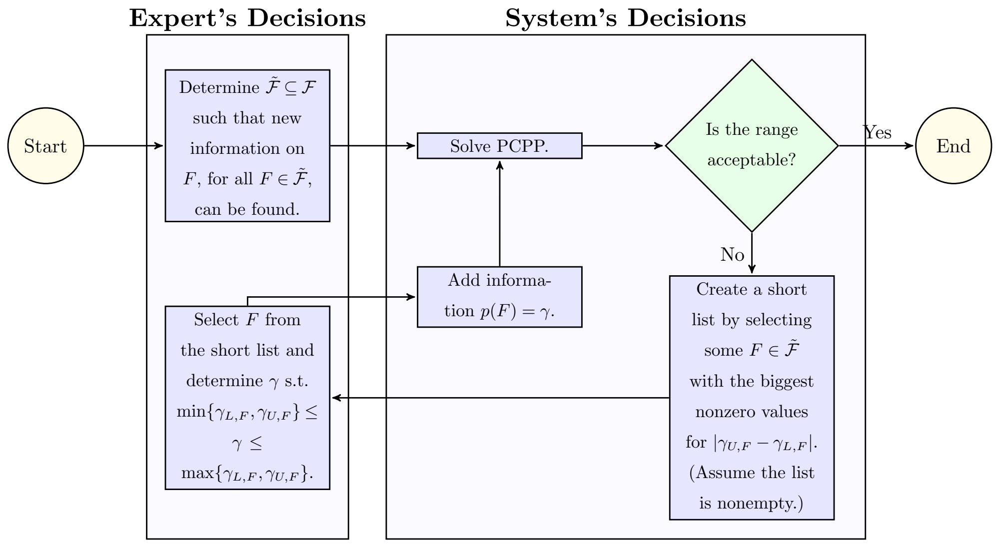

Figure 7 An interactive method for obtaining an acceptable range for a part's requirement.

The short lists are sorted in the non-increasing order of the width of the allowable intervals. We assume that the expert selects a formula and assigns the midpoint of its allowable interval to its penetration rate. It is seen that for part R_1, after two rounds of adding new information set and resolving the problem, the obtained range satisfies our criteria.

Step-by-step execution of our interactive method for part R_1

(1) The expert determines $\tilde{\mathcal{F}}$ as the set of all conjunctions of one or two positive options.

(2) Solving PCPP results in:
$2.155 \leq Q_{R_1} \leq 2.585$
The width of this range is 19.95% of the minimum value.

(3) The range is not acceptable.

(4) Short list:
$0.246 \leq p(A_1 \wedge A_4) \leq 0.347$
$0.289 \leq p(A_1 \wedge A_2) \leq 0.385$
$0.904 \leq p(A_2) \leq 1$
$0.289 \leq p(A_2 \wedge A_9) \leq 0.385.$

(5) The expert adds the following information set:
$p(A_2) = 0.952.$

(6) Solving PCPP results in:
$2.251 \leq Q_{R_1} \leq 2.481.$
The width of this range is 10.21% of the minimum value.

(7) The range is not acceptable.

(8) Short list:
$0.021 \leq p(A_5 \wedge A_9) \leq 0.096$
$0.027 \leq p(A_4 \wedge A_5) \leq 0.077$
$0.048 \leq p(A_1 \wedge A_5) \leq 0.096$
$0 \leq p(A_2 \wedge A_5) \leq 0.048$
$0.048 \leq p(A_5) \leq 0.096.$

(9) The expert adds the following information set:
$p(A_5 \wedge A_9) = 0.0585.$
In order to add this information set, we define an artificial option A' such that $p_{A'} = 0.0585$. Moreover, we add the following rule to the problem:
$A' \Leftrightarrow A_5 \wedge A_9.$

(10) Solving the PCPP results in:
$2.251 \leq Q_{R_1} \leq 2.439.$
The width of this range is 8.35% of the minimum value.

(11) The range is acceptable, and hence we stop!

Figure 8 Illustrating our interactive method (also see Example 2 and Fig. 2).

Appendix D: Supporting Material for Section 5

D.1. An example for the infeasibility of the point/range estimate found by the current approach

Consider options: A', A'', and A'''. Rules: $A''' \Rightarrow A' \vee A''$, $A' \Rightarrow A'''$, $A'' \Rightarrow A'''$. Condition code: $A' \wedge A'' \rightarrow R$. Penetration rates: $p_{A'} = \frac{1}{2}$, $p_{A''} = \frac{1}{2}$, and $p_{A'''} = \frac{4}{5}$.

The options that are related to part R are A' and A''. Producible sub-configurations: $\mathbf{y} = (y_{A'}, y_{A''}) \in \{(0,0),(1,0),(0,1),(1,1)\}$. The penetration rates $(p_{A'}, p_{A''}) = (\frac{1}{2}, \frac{1}{2})$ map into a set of points in the space of sub-configurations that has the following two extreme points: $(0, \frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, 0, \frac{1}{2})$. These extreme points map into requirement 0 and $\frac{1}{2}$, respectively. Thus, the current approach finds a point estimate in $[0, \frac{1}{2}]$ and then creates a range estimate using the 10% rule.

We next show that the obtained range $[0, \frac{1}{2}]$ contains only one feasible point as the requirement of R and any other point in this range is infeasible. Note that the set of rules can be summarized as $A''' \Leftrightarrow A' \vee A''$, which implies that $p_{A'''} = p(A' \vee A'') = \frac{4}{5}$. On the other hand, we have: $Q_R = p(A' \wedge A'') = p_{A'} + p_{A''} - p(A' \vee A'') = \frac{1}{5}$. Therefore, the only feasible point in the range $[0, \frac{1}{2}]$ is $\frac{1}{5}$. In addition, if the current approach determines, for example, $\frac{1}{2}$ as the point estimate, then the range estimate using the 10% rule will be $[0.45, 0.55]$. It is seen that the range estimate found by the current approach does not include the correct requirement of $\frac{1}{5}$.

Appendix E: Supporting Material for Section 6

E.1. Proof of Theorem 2: Generalization of Nonlinear Programming Equivalence

Define vector $\beta = (\beta_1, \ldots, \beta_n)$ as follows: $\beta_i := \sum_{(i,\alpha)\in\tilde{\mathscr{C}}(R)} \alpha$, for all $i = 1, \ldots, n$. Note that $\beta \in \mathbb{R}^n_+$ and recall that in this paper we assume $\mathbf{0} \in \mathbb{Y}$. Note that our problem (PG) can be equivalently formulated as follows:

$$\text{(PG):} \quad \min/\max \;\; \beta^\top Y\mathbf{x} \tag{24}$$

$$\text{s.t.} \quad \mathbf{p}_L \le Y\mathbf{x} \le \mathbf{p}_U, \tag{25}$$

$$\mathbf{1}^\top\mathbf{x} = 1, \;\; \mathbf{x} \in \mathbb{R}^m_+, \tag{26}$$

where $Y := [\mathbf{y}^1|\mathbf{y}^2|\ldots|\mathbf{y}^m]$, $m := |\mathbb{Y}|$, $\mathbf{p}_L := (p_{L,1}, \ldots, p_{L,n})$, and $\mathbf{p}_U := (p_{U,1}, \ldots, p_{U,n})$. We want to prove that, for sufficiently large M, solving the following problem provides an optimal solution to problem (PG).

$$\text{(PG}'\text{):} \quad \min/\max \;\; \beta^\top Y\mathbf{x} \pm M\sum_{i=1}^{n} \max\{p_{L,i} - Y_i\mathbf{x}, 0, Y_i\mathbf{x} - p_{U,i}\} \tag{27}$$

$$\text{s.t.} \quad \mathbf{1}^\top\mathbf{x} = 1, \;\; \mathbf{x} \in \mathbb{R}^m_+, \tag{28}$$

where Y_i denotes the ith row of matrix Y.

Let $\lambda_{\min}(n)$ denote the smallest eigenvalue of all matrices in the form of $B^\top B$ where B is an invertible matrix of size $(2n+1) \times (2n+1)$ with all entries being a member of $\{-1, 0, 1\}$. Since $B^\top B$ is symmetric and positive definite and there are a finite number of possibilities for B (and, consequently, for $B^\top B$), then $\lambda_{\min}(n)$ exists and $\lambda_{\min}(n) > 0$. Note that there are at most $3^{(2n+1)^2}$ possibilities for B (because $B \in \{-1,0,1\}^{(2n+1)\times(2n+1)}$ and B must be invertible); hence, one needs to find the eigenvalues of $B^\top B$ for each possibility, which results in at most $(2n+1)3^{(2n+1)^2}$ different values, and finally, select the smallest value.

We aim to prove that solving (PG′) provides an optimal solution to (PG) if:

$$M > \|\beta\|_1 \left(1 + \frac{2n+1}{\sqrt{\lambda_{\min}(n)}}\right).$$

Our proof is based on analyzing the impact of changing $\mathbf{p}_L$ and $\mathbf{p}_U$ on the optimal value of problem (PG). We only consider vectors $\mathbf{p}_L$ and $\mathbf{p}_U$ for which the feasible region of (PG) is nonempty. Define $\mathbb{P} := \{(\mathbf{p}_L, \mathbf{p}_U) \in [0,1]^{2n} |$ Problem (PG) has a feasible solution$\}$. Hence, for all $(\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P}$, problem (PG) has a bounded optimal value since: $0 \le \beta^\top \mathbf{p}_L \le \beta^\top Y\mathbf{x} \le \beta^\top \mathbf{p}_U \le \|\beta\|_1$ (also because $\beta \in \mathbb{R}^n_+$). Let the function $z(\mathbf{p}_L, \mathbf{p}_U)$, with domain $\mathbb{P}$, denote the optimal value of (PG). Thus, we have:

$$0 \le z(\mathbf{p}_L, \mathbf{p}_U) \le \|\beta\|_1, \quad \forall (\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P}.$$

We first focus on the maximization case of problem (PG). Dual of the maximization problem is as follows:

$$\text{(Dmax):} \quad \min \quad \mathbf{e_1}^\top \mathbf{p}_U - \mathbf{e_2}^\top \mathbf{p}_L + e_3 \tag{29}$$

$$\text{s.t.} \quad Y^\top \mathbf{e_1} - Y^\top \mathbf{e_2} + e_3 \mathbf{1} \ge Y^\top \beta, \tag{30}$$

$$\mathbf{e_1},\ \mathbf{e_2} \in \mathbb{R}^n_+, \tag{31}$$

where $\mathbf{e_1}$, $\mathbf{e_2}$, and e_3 are the dual variables (note that $e_3 \in \mathbb{R}$). The following lemma presents a characterization of the extreme points of the feasible set of the dual problem.

LEMMA 4. *Let $\mathbb{E}$ denote the set of extreme points of the feasible set of problem (Dmax). Then, (a) $1 \le |\mathbb{E}| < \infty$ and (b) $\|\mathbf{e}\|_1 \le \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right)$, for all $\mathbf{e} \in \mathbb{E}$.*

PROOF. (a) Eq. (30) consists of a constraint associated with each row of matrix $Y^\top$. Thus, since $\mathbf{0} \in \mathbb{Y}$, there exists a constraint in the form of $e_3 \ge 0$. Therefore, all dual variables are nonnegative. Define $\mathbf{e} := (\mathbf{e_1}, \mathbf{e_2}, e_3) \in \mathbb{R}^{2n+1}_+$ as the vector of dual variables. Note that the feasible set of (Dmax) does not contain a line because it is a subset of $\mathbb{R}^{2n+1}_+$. Moreover, the feasible set of (Dmax) is nonempty because (PG) has a bounded optimal value for all $(\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P}$ and $\mathbb{P}$ is nonempty (note that $(\mathbf{0}, \mathbf{1}) \in \mathbb{P}$). Thus, the feasible set of (Dmax) has at least one extreme point (due to Proposition 2.1.2 of Bertsekas (2009)). Additionally, a polyhedron (with at least one extreme point) that is defined by a finite number of (in)equalities have a finite number of extreme points.

(b) Note that the feasible set of (Dmax) is defined by $2n+1$ variables and $2n+m$ inequalities. Hence, an extreme point corresponds to $2n+1$ linearly independent binding (active) constraints. Therefore, an extreme point is obtained by finding $2n+1$ linearly independent constraints, setting them as equality, and solving the $2n+1$ equations which results in a unique solution $\mathbf{e}$; if $\mathbf{e}$ is feasible for problem (Dmax), then it is an extreme point. Let n' denote the number of linearly independent rows (or columns) of $[Y^\top \;\; -Y^\top \;\; \mathbf{1}]$ and note that $n' \le n+1$. Let us select $2n+1$ linearly independent constraints and set them as equality. This includes n'' constraints from Eq. (30), such that $n'' \le n'$, and $2n+1-n''$ constraints from Eq. (31). Denote the resulting system as $\begin{bmatrix}\bar{Y}_1\\ \bar{Y}_2\end{bmatrix}\mathbf{e} = \begin{bmatrix}\bar{Y}_1\\ \mathbf{0}\end{bmatrix}\binom{\beta}{\mathbf{0}}$ where $\bar{Y}_1$ and $\bar{Y}_2$ are matrices of size $n''\times(2n+1)$ and $(2n+1-n'')\times(2n+1)$ that correspond to the constraints that are selected from Eqs. (30) and (31), respectively, and $\mathbf{0}$ is a matrix (or vector) of appropriate size with all entries equal to zero. Note that:

$$\begin{bmatrix}\bar{Y}_1\\ \bar{Y}_2\end{bmatrix}\mathbf{e} = \begin{bmatrix}\bar{Y}_1\\ \mathbf{0}\end{bmatrix}\begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} + \begin{bmatrix}\bar{Y}_1\\ \bar{Y}_2\end{bmatrix}\begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} - \begin{bmatrix}\bar{Y}_1\\ \bar{Y}_2\end{bmatrix}\begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} = \begin{bmatrix}\bar{Y}_1\\ \bar{Y}_2\end{bmatrix}\begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} - \begin{bmatrix}\mathbf{0}\\ \bar{Y}_2\end{bmatrix}\begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix}.$$

Additionally, since $\begin{bmatrix}\bar{Y}_1\\ \bar{Y}_2\end{bmatrix}$ is invertible, we define $\bar{\bar{Y}} := -\begin{bmatrix}\bar{Y}_1\\ \bar{Y}_2\end{bmatrix}^{-1}$; hence, we have:

$$\begin{aligned}
\mathbf{e} &= \begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} + \bar{\bar{Y}}\begin{bmatrix}\mathbf{0}\\ \bar{Y}_2\binom{\beta}{\mathbf{0}}\end{bmatrix}\\
&\le \begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} + \left\|\bar{\bar{Y}}\begin{bmatrix}\mathbf{0}\\ \bar{Y}_2\binom{\beta}{\mathbf{0}}\end{bmatrix}\right\|_2 \mathbf{1}\\
&= \begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} + \left(\begin{bmatrix}\mathbf{0}\\ \bar{Y}_2\binom{\beta}{\mathbf{0}}\end{bmatrix}^\top \bar{\bar{Y}}^\top\bar{\bar{Y}}\begin{bmatrix}\mathbf{0}\\ \bar{Y}_2\binom{\beta}{\mathbf{0}}\end{bmatrix}\right)^{1/2}\mathbf{1}\\
&\le \begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} + \sqrt{1/\lambda_{\min}(n)}\left\|\begin{bmatrix}\mathbf{0}\\ \bar{Y}_2\binom{\beta}{\mathbf{0}}\end{bmatrix}\right\|_2 \mathbf{1}\\
&\le \begin{pmatrix}\beta\\ \mathbf{0}\end{pmatrix} + \sqrt{1/\lambda_{\min}(n)}\,\|\beta\|_1\,\mathbf{1}.
\end{aligned}$$

Note that the fourth line follows because $1/\lambda_{\min}(n)$ is an upper bound on the eigenvalues of $\bar{\bar{Y}}^\top\bar{\bar{Y}}$. Additionally, since $\mathbf{e}\ge\mathbf{0}$, then we have:

$$\|\mathbf{e}\|_1 \le \|\beta\|_1\left(1+(2n+1)\sqrt{1/\lambda_{\min}(n)}\right),$$

and hence the proof is complete. □

Using Lemma 4, the feasible set of (Dmax) has at least one extreme point. Moreover, since the optimal value of (Dmax) is attained for all $(\mathbf{p}_L,\mathbf{p}_U)\in\mathbb{P}$, then an extreme point of the feasible set of (Dmax) is optimal (due to Proposition 2.4.1 of Bertsekas (2009)). Therefore,

$$z(\mathbf{p}_L,\mathbf{p}_U) = \min_{\mathbf{e}\in\mathbb{E}}\ \mathbf{e}^\top(\mathbf{p}_U,-\mathbf{p}_L,1), \qquad \forall(\mathbf{p}_L,\mathbf{p}_U)\in\mathbb{P}.$$

It is easy to see that z is a piecewise linear and concave function over $\mathbb{P}$. Consider changing $(\mathbf{p}_L,\mathbf{p}_U)$ to $(\mathbf{p}_L-\mathbf{\Delta_1},\mathbf{p}_U+\mathbf{\Delta_2})\in\mathbb{P}$ such that $\mathbf{\Delta_1}$ and $\mathbf{\Delta_2}$ are arbitrary vectors in $\mathbb{R}^n_+$. Since z is concave, then:

$$z(\mathbf{p}_L-\mathbf{\Delta_1},\mathbf{p}_U+\mathbf{\Delta_2}) \le z(\mathbf{p}_L,\mathbf{p}_U) + \nabla z(\mathbf{p}_L,\mathbf{p}_U)^\top(-\mathbf{\Delta_1},\mathbf{\Delta_2}),$$

where ∇z denotes the subgradient of z, which is unique and equal to $(-\mathbf{e}_2, \mathbf{e}_1)$ for some $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}$ if z is linear at $(\mathbf{p}_L, \mathbf{p}_U)$. Note that if there are more than one $\mathbf{e} \in \mathbb{E}$ that achieve minimum at $(\mathbf{p}_L, \mathbf{p}_U)$, then z is not linear and its subgradient is the convex hull of all such $(-\mathbf{e}_2, \mathbf{e}_1)$'s. In addition, the above inequality holds for all $(-\mathbf{e}_2, \mathbf{e}_1)$ where $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}$ achieves the minimum at $(\mathbf{p}_L, \mathbf{p}_U)$. Thus, there exists $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}$ such that:

$$\begin{aligned} z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) &\leq z(\mathbf{p}_L, \mathbf{p}_U) + (-\mathbf{e}_2, \mathbf{e}_1)^\top (-\mathbf{\Delta_1}, \mathbf{\Delta_2}) \\ &\leq z(\mathbf{p}_L, \mathbf{p}_U) + \|\mathbf{e}\|_1 \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \\ &\leq z(\mathbf{p}_L, \mathbf{p}_U) + \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right). \end{aligned}$$

Let $\mathbf{x}$ denote an arbitrary feasible solution of problem (PG$'$). Define $\Delta_{1,i} := \max\{0, p_{L,i} - Y_i\mathbf{x}\}$ and $\Delta_{2,i} := \max\{0, Y_i\mathbf{x} - p_{U,i}\}$, for all i, and let $\mathbf{\Delta_1} := (\Delta_{1,1}, \ldots, \Delta_{1,n})$ and $\mathbf{\Delta_2} := (\Delta_{2,1}, \ldots, \Delta_{2,n})$. In addition, assume that at least one of $\mathbf{\Delta_1}$ or $\mathbf{\Delta_2}$ is nonzero implying that $\mathbf{x}$ is an infeasible point for problem (PG). We have:

$$z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) - \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right) \leq z(\mathbf{p}_L, \mathbf{p}_U).$$

Let $z'(\mathbf{x})$ denote the objective value of problem (PG$'$) for the feasible solution $\mathbf{x}$. Let $\mathbf{x}^*$ denote the optimal solution of problem (PG). Note that $\mathbf{x}^*$ is feasible for problem (PG$'$). We next show that $z'(\mathbf{x}) < z'(\mathbf{x}^*)$, which implies that $\mathbf{x}^*$ is an optimal solution of (PG$'$).

$$\begin{aligned} z'(\mathbf{x}) &= \beta^\top Y\mathbf{x} - M\sum_{i=1}^{n} \max\{p_{L,i} - Y_i\mathbf{x}, 0, Y_i\mathbf{x} - p_{U,i}\} \\ &\leq z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) - M\|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \\ &< z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) - \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right) \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \\ &\leq z(\mathbf{p}_L, \mathbf{p}_U). \end{aligned}$$

The second line follows because $\beta^\top Y\mathbf{x} \leq z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2})$ (here, note that $\mathbf{x}$ is a feasible solution of (PG) after we change $(\mathbf{p}_L, \mathbf{p}_U)$ to $(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2})$ while $z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2})$ is the optimal value after this change). The third line follows because we assume $M > \|\beta\|_1 \left(1 + (2n+1)\sqrt{1/\lambda_{\min}(n)}\right)$. Finally, we observe that:

$$z'(\mathbf{x}) < z(\mathbf{p}_L, \mathbf{p}_U) = z'(\mathbf{x}^*),$$

which completes the proof for the maximization case of problem (PG). The proof for the minimization case is very similar. Dual of the minimization problem is as follows:

$$\text{(Dmin):} \quad \max \quad -\mathbf{e_1}^\top \mathbf{p}_U + \mathbf{e_2}^\top \mathbf{p}_L - e_3 \tag{32}$$

$$\text{s.t.} \quad -Y^\top \mathbf{e_1} + Y^\top \mathbf{e_2} - e_3\mathbf{1} \leq Y^\top \beta, \tag{33}$$

$$\mathbf{e_1}, \ \mathbf{e_2} \in \mathbb{R}^n_+, \tag{34}$$

The following lemma presents a characterization of the extreme points of (Dmin).

LEMMA 5. *Let $\mathbb{E}_{\min}$ denote the set of extreme points of the feasible set of problem (Dmin). Then, (a) $1 \le |\mathbb{E}_{\min}| < \infty$ and (b) $\|\mathbf{e}\|_1 \le \|\beta\|_1 \left(1+(2n+1)\sqrt{1/\lambda_{\min}(n)}\right)$, for all $\mathbf{e} \in \mathbb{E}_{\min}$.*

PROOF. The proof follows the same steps as the proof of Lemma 4. The main difference is that in part (b), when we select $2n+1$ linearly independent constraints, the resulting system is denoted as $\begin{bmatrix} \bar{Y}_1 \\ \bar{Y}_2 \end{bmatrix} \mathbf{e} = \begin{bmatrix} \bar{Y}_1 \\ \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \beta \\ 0 \end{bmatrix}$. The rest of the proof is similar and hence is skipped. □

Using Lemma 5, we can formulate the minimization case of (PG) as:

$$z_{\min}(\mathbf{p}_L, \mathbf{p}_U) = \max_{\mathbf{e} \in \mathbb{E}_{\min}} \mathbf{e}^\top(-\mathbf{p}_U, \mathbf{p}_L, -1), \qquad \forall (\mathbf{p}_L, \mathbf{p}_U) \in \mathbb{P},$$

where $z_{\min}(\mathbf{p}_L, \mathbf{p}_U)$ denotes the optimal value of the minimization case of (PG) and is a piecewise linear and convex function over $\mathbb{P}$. Consider changing $(\mathbf{p}_L, \mathbf{p}_U)$ to $(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) \in \mathbb{P}$ such that $\mathbf{\Delta_1}$ and $\mathbf{\Delta_2}$ are arbitrary vectors in $\mathbb{R}^n_+$. Since z is convex, then there exists $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, e_3) \in \mathbb{E}_{\min}$ such that:

$$\begin{aligned} z(\mathbf{p}_L - \mathbf{\Delta_1}, \mathbf{p}_U + \mathbf{\Delta_2}) &\ge z(\mathbf{p}_L, \mathbf{p}_U) + (\mathbf{e}_2, -\mathbf{e}_1)^\top(-\mathbf{\Delta_1}, \mathbf{\Delta_2}) \\ &\ge z(\mathbf{p}_L, \mathbf{p}_U) - \|\mathbf{e}\|_1 \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \\ &\ge z(\mathbf{p}_L, \mathbf{p}_U) - \|(\mathbf{\Delta_1}, \mathbf{\Delta_2})\|_1 \|\beta\|_1 \left(1+(2n+1)\sqrt{1/\lambda_{\min}(n)}\right). \end{aligned}$$

The rest of the proof for the minimization case of (PG) is very similar to the maximization case that we showed above, and hence, is skipped. Thus, the proof is complete.

E.2. Proof of Proposition 4: Requirement for a Group of Parts

We have:

$$\begin{aligned} \sum_{R \in \tilde{\mathcal{R}}} Q_R &= \sum_{R \in \tilde{\mathcal{R}}} \sum_{(F,\alpha) \in \mathscr{C}(R)} \alpha p(F) \\ &= \sum_{R \in \tilde{\mathcal{R}}} \sum_{F \in \mathscr{C}\mathscr{F}(R)} p(F) \sum_{\alpha:(F,\alpha) \in \mathscr{C}(R)} \alpha \\ &= \sum_{R \in \tilde{\mathcal{R}}} \sum_{F \in \bigcup_{\bar{R} \in \tilde{\mathcal{R}}} \mathscr{C}\mathscr{F}(\bar{R})} p(F) \sum_{\alpha:(F,\alpha) \in \mathscr{C}(R)} \alpha \\ &= \sum_{F \in \bigcup_{\bar{R} \in \tilde{\mathcal{R}}} \mathscr{C}\mathscr{F}(\bar{R})} p(F) \sum_{R \in \tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha) \in \mathscr{C}(R)} \alpha \\ &= \sum_{\substack{F \in \bigcup_{\bar{R} \in \tilde{\mathcal{R}}} \mathscr{C}\mathscr{F}(\bar{R}) \\ \alpha' = \sum_{R \in \tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha) \in \mathscr{C}(R)} \alpha}} \alpha' p(F) \\ &= \sum_{(F,\alpha') \in \mathscr{C}(\hat{R})} \alpha' p(F), \end{aligned}$$

where, $\mathscr{C}(\hat{R}) = \{(F,\alpha') | F \in \bigcup_{R \in \tilde{\mathcal{R}}} \mathscr{C}\mathscr{F}(R), \alpha' = \sum_{R \in \tilde{\mathcal{R}}} \sum_{\alpha:(F,\alpha) \in \mathscr{C}(R)} \alpha\}$. Note that the third line follows because if $F \notin \mathscr{C}\mathscr{F}(R)$, then $\sum_{\alpha:(F,\alpha) \in \mathscr{C}(R)} \alpha = 0$. Hence, the proof is complete.

E.3. Illustration of finding a range for a group of parts

We illustrate Proposition 4 on parts R_1, R_2, and R_3, that we introduced in Example 2. Our approach is graphically shown in Fig. 9.

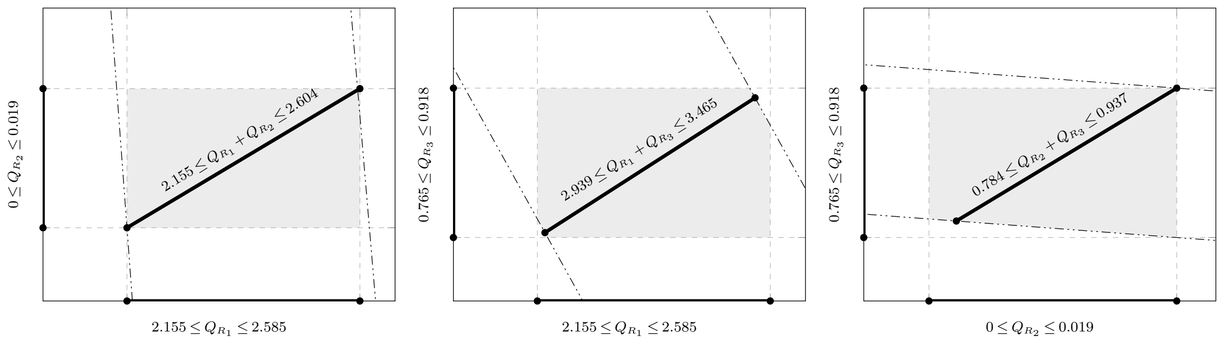

Figure 9 Graphical illustration of finding concurrent ranges for parts.

To show the effectiveness of the range obtained using Proposition 4, we compare it to a simple range on the requirement of a group of parts that is obtained by simply summing the minimum and maximum values on the ranges for individual parts. This simple approach results in the following ranges:

$$2.155 \le Q_{R_1} + Q_{R_2} \le 2.604$$
$$2.92 \le Q_{R_1} + Q_{R_3} \le 3.503$$
$$0.765 \le Q_{R_2} + Q_{R_3} \le 0.937$$
$$2.92 \le Q_{R_1} + Q_{R_2} + Q_{R_3} \le 3.522.$$

We next obtain these ranges using Proposition 4. Consider finding a range for $(Q_{R_1} + Q_{R_2})$. Define a hypothetical part $\hat{R}$ with the following set of condition codes:

$$\begin{aligned}\mathscr{C}(\hat{R}) = \ & \{(A_1, 1),\ ((A_2 \wedge \neg A_3) \vee (A_2 \wedge A_4), 2), \\ & ((\neg A_5 \wedge A_7), 4),\ ((A_3 \wedge A_4) \vee (A_3 \wedge A_6), 1)\}.\end{aligned}$$

Note that in this example, $\mathscr{C}(\hat{R})$ is simply the union of $\mathscr{C}(R_1)$ and $\mathscr{C}(R_2)$. We obtain the following range for $Q_{\hat{R}}$, which is in fact a range for $(Q_{R_1} + Q_{R_2})$.

$$2.155 \le Q_{\hat{R}} \le 2.604.$$

Thus, by using Proposition 4, and by repeating the above procedure for other combinations, we obtain the following ranges:

$$2.155 \le Q_{R_1} + Q_{R_2} \le 2.604$$

$$2.939 \le Q_{R_1} + Q_{R_3} \le 3.465$$

$$0.784 \le Q_{R_2} + Q_{R_3} \le 0.937$$

$$2.939 \le Q_{R_1} + Q_{R_2} + Q_{R_3} \le 3.484.$$

In this example, the range on the group of parts is about 90% of the range obtained by summing the ranges for individual parts.

Appendix F: Notations

Abbreviations:

CPS	Configurations-level penetration statistic
GAM	Global auto manufacturer
MILP	Mixed-integer linear programming
OPS	Options-level penetration statistic
P1, P1min, P1max	Our MILP problems
P2, P2L, P2U	Our NLP problems
PCPP	Parts capacity-planning problem (refers to (P1) and/or (P2))

Notations:

$A, A_1, A_2, \dots \in \mathcal{N}$	Options ($\mathcal{N}$ is the set of all options)
$R, R_1, R_2, \dots \in \mathcal{R}$	Parts ($\mathcal{R}$ is the set of all parts)
$F, F_1, F_2, \dots \in \mathcal{F}$	Propositional formulas ($\mathcal{F}$ is the set of all propositional formulas)
$\mathscr{C}(R)$, $\tilde{\mathscr{C}}(R)$	The set of original and simplified condition codes for part R, respectively
$v_{\mathbf{y}}(F)$	The value of formula F in configuration $\mathbf{y}$
p	Penetration rate (defined for both formulas and options)
Q_R	Requirement of part R
$i = 1, \dots, n$	Index for options
$\mathbf{y} = (y_1, \dots, y_n)$	A producible configuration
$a_{\mathbf{y}}$	Penetration rate of configuration $\mathbf{y}$
$\mathbb{Y}$, $conv(\mathbb{Y})$	The set of all producible configurations and its convex hull
M	A sufficiently big number
$\mathbf{w}_R$, $\mathbf{Diag}(\mathbf{w}_R)$	The weight vector and the corresponding diagonal matrix in (P2)
$\hat{\mathbf{p}}_{R,L}$, $\hat{\mathbf{p}}_{R,U}$	Parameters defined and used in formulating (P2)
$Q^*_{R,L}$, $Q^*_{R,U}$	Optimal values of the PCPP (min. and max., respectively)
k	Iteration counter in our algorithm
$\mathbf{y}^{(k)}$	The new extreme point of $conv(\mathbb{Y})$ generated at iteration k
$\mathbf{p}^{(k)}$	The best known solution at iteration k of our algorithm
$\mathbf{g}^{(k)}$	The subgradient at iteration k of our algorithm
$\mathcal{N}_1$, n_1	The set and the number of options with forecast penetration rates, respectively
γ	The right-hand-side value of an information $p(F) = \gamma$